Maximum Achievable Throughputs for Uncoded OPPM and MPPM in Optical Direct-Detection Channels

Hossam M. H. Shalaby, *Member, IEEE*

Abstract— **Tight upper and lower bounds on the maximum throughputs for both overlapping and multipulse pulse position modulation (OPPM and MPPM) in optical direct-detection channels are derived. The corresponding pulse-position multiplicities are estimated. Simple lower bounds for the maximum achievable throughput (with the error rate not exceeding a certain threshold) are obtained for OPPM. A comparison between the maximum achievable throughputs for both OPPM and MPPM is also considered. Our results suggest using MPPM for high efficient transmission and OPPM for low efficient transmission. Namely, MPPM should be used when the transmission efficiency exceeds 0.027 nats/photon.**

I. Introduction

BECAUSE of its tremendous prosperity in direct-detection optical channels, the pulse-position modulation (PPM) signaling format has been the modulation remedy for many low-rate direct-detection applications. PPM, however, is not convenient for high data rate applications because in this case the laser pulsewidth should be decreased in order to achieve the required throughput. Recently, it has been shown that new modulation schemes, such as optical overlapping pulse-position modulation (OPPM) [1]–[6] and multipulse pulse-position modulation (MPPM) [7]–[9], can offer large throughput without the need to decrease the laser pulsewidth. Of course this advantage is acquired at the expense of a large cost in the error rate performance of both OPPM and MPPM. Moreover, a tough synchronization requirement is imperative for OPPM with large overlapping index. Fortunately, an improvement in OPPM or MPPM performance can be fulfilled by employing error correcting codes and sacrificing some of the throughput gain [3], [6]. In a recent paper by Georghiades [6], the above modulation schemes have been investigated and compared in terms of the capacity, cutoff rate, and error-probability performance of uncoded and trellis-coded systems. Our main goal in the following is to distinguish between the regions of preference of OPPM and MPPM under a constraint on the bit error rate.

This paper can principally be divided into two main parts. In the first part we aim at obtaining simple and tight lower bounds on the achievable throughputs for both uncoded OPPM and MPPM. Our second aim is to determine the best throughputs that can be achieved for both OPPM and MPPM under an upper bound on the probability of error. To have more insight on the results obtained, we restrict our study to quantum-limited direct-detection optical channels only. Our results suggest using MPPM for high efficient transmission (exceeding 0.027 nats/photon) and using OPPM for low efficient transmission.

The paper is organized as follows: In Section II we give a description of an OPPM channel model. We also provide tight upper and lower bounds on the maximum throughput for OPPM. Simple lower bounds for the maximum achievable throughput of OPPM (with the error rate not exceeding a certain threshold) are obtained in Section III. In Section IV we describe the channel model of MPPM and provide some bounds on the throughputs. A comparison between the maximum achievable throughputs for both OPPM and MPPM is also included in this section. Finally some remarks and conclusions are given in Section V.

Manuscript received August 29, 1994; revised March 24, 1995.

The author is with the Department of Electrical Engineering, Faculty of Engineering, University of Alexandria, Egypt.

IEEE Log Number 9414925.

II. Overlapping Pulse-Position Modulation

A. OPPM Channel Model

The channel model for direct-detection optical OPPM with overlapping index $N \in \{1, 2, \cdots\}$ was introduced in [2]. In OPPM the information is conveyed by the position of a laser pulse of duration τ within a time frame of width T. An overlap with depth $\left(1 - \frac{1}{N}\right)\tau$ is allowed between any two adjacent positions. We assume that there are M possible positions within the time frame. The transmitted pulse is said to be in position x, $x \in \{1, 2, \cdots, M\}$, if it extends over the subinterval starting at time $(x-1)\frac{\tau}{N}$ and ending τ s later. This subinterval will be called a slot. It is obvious that each slot is subdivided into N smaller subintervals of width τ/N, each will be called a chip. The relation between T, N, M, and τ is thus

$$T = (M + N - 1)\frac{\tau}{N}.$$

Let the channel input (transmitted pulse position) be denoted by the random variable X and the channel output be denoted by Y. The possible channel outputs and the transition probabilities $P_{Y|X}$ can be found in [2]. We denote by s the probability that the OPPM pulse is not erased. Thus

$$s \overset{def}{=} 1 - \exp[-Q] \, , \tag{1}$$

where Q represents the average photon count per pulse. If ρ denotes the transmitted information in nats/photon, then

$$Q = \frac{\log M}{\rho} . \tag{2}$$

B. Error Probability for Uncoded OPPM

The average symbol error probability $P_E(N)$ for uncoded OPPM was derived in [3]. We provide here tight bounds on this error rate. $P_E(N)$, for equally likely *input* data symbols, can be written as

$$P_E(N) = \sum_{x=1}^{M} \frac{1}{M} \Pr\{E|X = x\} . \tag{3}$$

Let $X = 1$. If chip 1 is erased, an event that occurs with probability $e^{-\frac{Q}{N}}$, then there is an ambiguity between at least two symbols. Thus we have a chance of at least 1/2 to choose the wrong symbol

$$\Pr\{E|X = 1\} \geq \frac{1}{2} e^{-\frac{Q}{N}} .$$

$\Pr\{E|X = M\}$ can be bounded in a similar way. If $X \in \{2, 3, \cdots, M-1\}$, then the ambiguity occurs when at least one chip out of the two chips 1 and N is erased. Hence

$$\Pr\{E|X = x\} \geq \frac{1}{2} e^{-\frac{Q}{N}} \left(2 - e^{-\frac{Q}{N}}\right), \; x \in \{2, 3, \cdots, M-1\} .$$

Substituting in (3) yields

$$P_E(N) \geq \frac{M-1}{M} e^{-\frac{Q}{N}} - \frac{M-2}{2M} e^{-2\frac{Q}{N}} \geq \frac{1}{2} e^{-\frac{Q}{N}} = \frac{1}{2} M^{-\frac{1}{N\rho}} . \tag{4}$$

On the other hand an upper bound on $P_E(N)$ can be obtained as follows. Assume that $X = 1(M)$, then we have a correct event if at least the received count in chip $1(M)$ is positive. Thus for $x \in \{1, M\}$, $\Pr\{C|X = x\} \geq 1 - e^{-\frac{Q}{N}}$ or

$$\Pr\{E|X = x\} \leq e^{-\frac{Q}{N}}, \qquad x \in \{1, M\}.$$

Similarly if $X \in \{2, 3, \cdots, M-1\}$, then the correct event occurs when at least the received photon counts in both chips 1 and N are not erased. Thus

$$\Pr\{E|X = x\} \leq 2e^{-\frac{Q}{N}} - e^{-2\frac{Q}{N}} \leq 2e^{-\frac{Q}{N}}, \quad x \in \{2, 3, \cdots, M-1\} .$$

Substituting in (3) yields

$$P_E(N) \leq 2\frac{M-1}{M} e^{-\frac{Q}{N}} \leq 2e^{-\frac{Q}{N}} = 2M^{-\frac{1}{N\rho}} . \tag{5}$$

The throughput of OPPM (in nats/slot) is given by

$$R(N, M) \stackrel{def}{=} \frac{\log M}{\frac{1}{N}(M + N - 1)} . \tag{6}$$

The maximum throughput for every N is given by

$$R_{\max}(N) \stackrel{def}{=} \max_M R(N, M) .$$

We are interested in estimating the optimum value of the pulse-position multiplicity M^* so as to have a maximum throughput. Define

$$f(M) \stackrel{def}{=} 1 + \frac{N-1}{M} - \log M .$$

It is obvious that M^* is the solution of the equation

$$\frac{\partial}{\partial M} R(N, M) = 0 \qquad \text{or} \qquad f(M) = 0 .$$

The following lemma gives an estimate of M^*.

Lemma 1: The optimum pulse-position multiplicity of the uncoded optical direct-detection OPPM channel with overlapping index $N \in \{1, 2, \cdots\}$ is upper bounded by

$$M^* \leq \frac{2N}{\log(N+1)}$$

and lower bounded by

$$M^* \geq \sqrt{N} + \frac{N}{\log(N+1)} .$$

Proof: It is easy to check that $f(M)$ is a decreasing function of M. Thus it suffices to show that

$$f\left(\frac{2N}{\log(N+1)}\right) \leq 0 \quad \text{and} \quad f\left(\sqrt{N} + \frac{N}{\log(N+1)}\right) \geq 0 .$$

Using Appendix A we can write

$$f(M) = 1 + \frac{N-1}{M} - \log M \leq 1 + \frac{N-1}{M} - a + \frac{e^{a-1}}{M} .$$

The last expression is nonpositive if

$$M \geq \frac{N - 1 + e^{a-1}}{a-1} = \frac{2N}{\log(N+1)} ,$$

where the last equality holds if we choose $a = 1 + \log(N+1)$. Whence $M^* \leq \frac{2N}{\log(N+1)}$.

The proof of the lower bound can be found in Appendix B. □

Theorem 1: In an uncoded optical direct-detection OPPM channel with overlapping index $N \in \{1, 2, \cdots\}$ the maximum throughput can be lower bounded as

$$R_{\max}(N) \geq \frac{\log\lceil \frac{2N}{\log(N+1)} \rceil}{1 + \frac{1}{N}\lceil \frac{2N}{\log(N+1)} \rceil - \frac{1}{N}}$$

and upper bounded as

$$R_{\max}(N) \leq \frac{\log\lceil \frac{2N}{\log(N+1)} \rceil}{1 + \frac{1}{N}\frac{N + \sqrt{N}\log(N+1)}{\log(N+1)} - \frac{1}{N}} ,$$

where $\lceil x \rceil$ denotes the smallest integer not less than x.

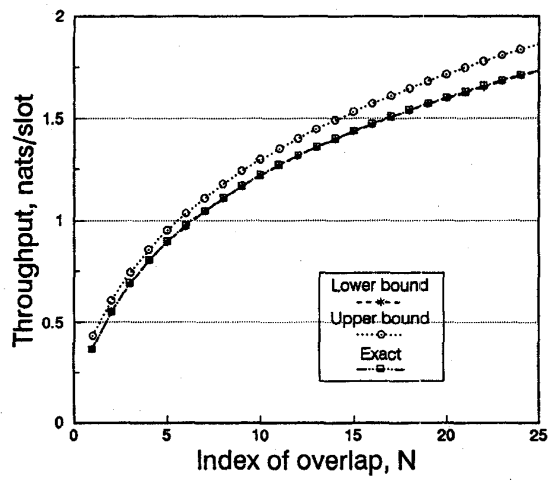

Fig. 1. Maximum OPPM throughput versus the index of overlap.

Proof: The lower bound is immediate by setting $M = \lceil \frac{2N}{\log(N+1)} \rceil$ in $R(N, M)$. Applying Lemma 1, the upper bound can be estimated as follows:

$$R_{\max}(N) = \frac{\log M^*}{\frac{1}{N}(M^* + N - 1)} \leq \frac{\log \lceil \frac{2N}{\log(N+1)} \rceil}{1 + \frac{1}{N} \frac{N+\sqrt{N}\log(N+1)}{\log(N+1)} - \frac{1}{N}} . \qquad \square$$

The upper and lower bounds are plotted in Fig. 1 along with the exact values of $R_{\max}(N)$. The tightness of the bounds is obvious from the figure. It is hard to find a gap between the lower bound and the exact values of the maximum throughput. This indicates that the optimum value of M is so close to $\lceil \frac{2N}{\log(N+1)} \rceil$.

The above theorem demonstrates that we can increase the throughput as we wish, but this will be accompanied by an increase in the error rate. In the remaining of this section we would like to examine the behavior of the throughput when setting a constraint on the error rate. We will require that $P_E(N) \leq \epsilon$. From (5) this requirement is assured whenever $M \geq (2/\epsilon)^{N\rho}$. For sake of convenience we define the parameter $t \overset{def}{=} (2/\epsilon)^{\rho}$. Thus we consider the following optimization problem:

$$R^*(N) \overset{def}{=} \max_{M: M \geq t^N} R(N, M) . \qquad (7)$$

The following theorem provides a tight lower bound to the above problem.

Theorem 2: In an uncoded optical direct-detection OPPM channel with overlapping index $N \in \{1, 2, \cdots\}$ the maximum throughput defined in (7) can be lower bounded as

$$R^*(N) \geq \begin{cases} R(N, \lceil t^N \rceil) ; & \text{if } t^N > \lceil \frac{2N}{\log(N+1)} \rceil , \\ R(N, \lceil \frac{2N}{\log(N+1)} \rceil) ; & \text{else.} \end{cases}$$

Proof: If $t^N \leq \lceil \frac{2N}{\log(N+1)} \rceil$, the optimum value of M^* is *almost* $\lceil \frac{2N}{\log(N+1)} \rceil$. If $t^N > \lceil \frac{2N}{\log(N+1)} \rceil$, the optimum value of M^* lies on the boundary of the feasible region (which is $\lceil t^N \rceil$) since $R(N, M)$ is decreasing in M as long as $M > M^*$. $\square$

The constraint throughput (given by Theorem 2) is plotted in Fig. 2 versus the overlapping index N and ρ for $\epsilon = 10^{-9}$. It is seen from the figure that for fixed ρ, the throughput increases with N until a certain value, after which it begins to decrease. This value is the maximum throughput that can be achieved given an efficiency (nats/photon) and an error rate constraint. It is obvious from the figure that this value increases as ρ decreases. In the following section we are interested in estimating the behavior of the maximum throughput that can be achieved for every ρ given an error rate constraint.

III. Maximum Achievable Throughput for OPPM

As mentioned previously, in this section we are interested in solving the optimization problem

$$R_N^* \overset{def}{=} \max_{\substack{M,N: \\ M \geq t^N}} R(N, M) . \qquad (8)$$

The next theorem provides a good approximate solution to the above problem.

Theorem 3: In an uncoded optical direct-detection OPPM channel with overlapping index $N \in \{1, 2, \cdots\}$ the maximum achievable throughput defined in (8) can be bounded as

$$R(N^*, \lceil t^{N^*} \rceil) \leq R_N^* \leq R(n^*, t^{n^*}) ,$$

where $N^* = \max\{\lceil n \rceil, 1\}$, $n^* = \max\{n, 1\}$, and n is the solution of the equation

$$t^n(2 - n \log t) + n - 2 = 0 .$$

Proof:

A. The Lower Bound

$$R_N^* = \max_{\substack{M,N: \\ M \geq t^N}} R(N, M) = \max_{N} \max_{\substack{M: \\ M \geq t^N}} R(N, M)$$
$$\geq \max_{N} R(N, \lceil t^N \rceil) \geq R(N^*, \lceil t^{N^*} \rceil) .$$

Indeed, the first inequality is true since we have chosen $M = \lceil t^N \rceil \geq t^N$ and the last inequality is true since $N^* \geq 1$.

B. The Upper Bound

We start by noticing that

$$R_N^* \leq \max\{R^1, R^2\} ,$$

where

$$R^1 \overset{def}{=} \max_{\substack{n \geq 1: \\ t^n \leq m_n}} R(n, m_n) , \qquad R^2 \overset{def}{=} \max_{\substack{n \geq 1: \\ t^n \geq m_n}} R(n, t^n) , \quad (9)$$

and m_n is the solution of $f(m_n) = 0$ or

$$1 + \frac{n-1}{m_n} - \log m_n = 0 . \qquad (10)$$

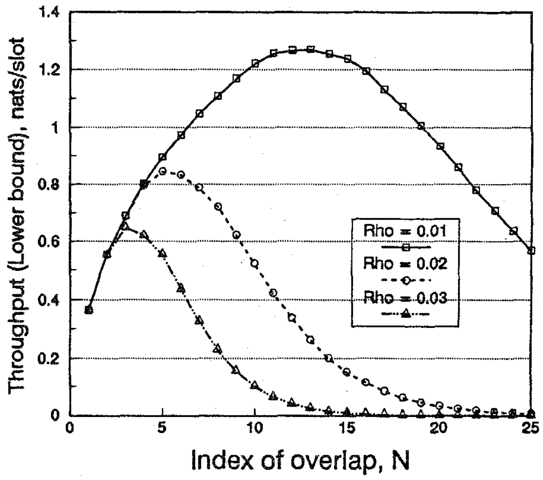

Fig. 2. Lower bounds on the maximum OPPM throughput (subject to $P_E \leq 10^{-9}$) versus both the index of overlap and the information efficiency in nats/photon.

m_n increases with n. Indeed, the last equation can be written as $m_n \log \frac{m_n}{e} = n-1$. But $x \log \frac{x}{e}$ is a nonnegative increasing function in x only if $x \geq e$. Whence $m_n \geq e$ and increases with n. This in turn implies that $R(n, m_n)$ is an increasing function in n. Indeed it is obvious from (10) that

$$\frac{m_n + n - 1}{m_n} = \log m_n \qquad \text{and} \qquad \frac{n}{m_n} = \log \frac{m_n}{e} + \frac{1}{m_n}$$

which yield

$$R(n, m_n) = \frac{n \log m_n}{n + m_n - 1} = \frac{n}{m_n} = \log \frac{m_n}{e} + \frac{1}{m_n} .$$

But $\log \frac{x}{e} + \frac{1}{x}$ is an increasing function in x for any $x \geq 1$. Thus we have shown that $R(n, m_n)$ increases with m_n. Since m_n increases with n, we obtain that $R(n, m_n)$ is an increasing function in n. Now from (9)

$$R^1 = \max_{\substack{n \geq 1: \\ t^n \leq m_n}} R(n, m_n) = \begin{cases} R(n_1, m_{n_1}) \,; & \text{if } t \leq e \,, \\ 0 \,; & \text{if } t > e \,, \end{cases}$$

where $n_1 \geq 1$ is the solution of the equation $t^{n_1} = m_{n_1}$. If $t > e$ a solution does not exist for $n_1 \geq 1$. This interprets the zero value of R^1. R^1 can thus be upper bounded as follows:

$$R^1 \leq \max_{n \geq 1} R(n, t^n) \,.$$

But R^2 has the same upper bound as above. Hence

$$R_N^* \leq \max_{n \geq 1} R(n, t^n) = \max_{n \geq 1} \frac{n \log t^n}{t^n + n - 1} \,.$$

To solve the last optimization problem we Differentiate the last function and equate the result to zero

$$t^n (2 - n \log t) + n - 2 = 0 \,.$$

If the solution of the above function $n \geq 1$, then $n^* = n$. Otherwise $n^* = 1$. □

Our last result in this section is to provide an estimate of N^* given in the previous theorem.

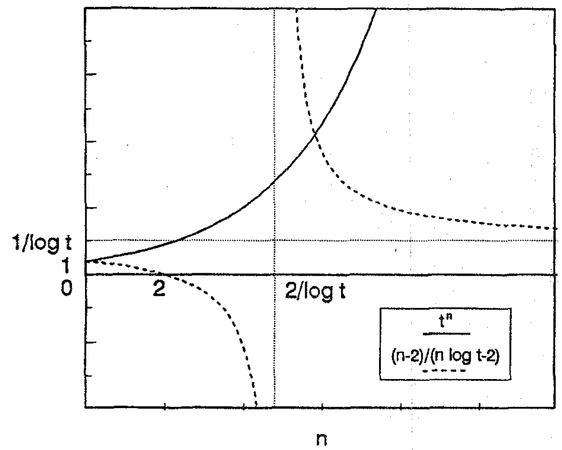

Fig. 3. Typical shapes of the functions t^n and $\frac{n-2}{n \log t - 2}$, where $\log t \leq 1$.

Proposition 1: If $\log t \leq 1$, then N^* given in Theorem 3 can be lower bounded by

$$N^* \geq \left\lceil \frac{\max\{2, \log \frac{1}{\log t}\}}{\log t} \right\rceil .$$

Remark: The motivation behind this proposition is that we seek the intersection of the two functions t^n and $\frac{n-2}{n \log t - 2}$. As seen in Fig. 3, the two extremes of the second function are the vertical asymptote $\frac{2}{\log t}$ and the horizontal asymptote $\frac{1}{\log t}$. The intersection of t^n with the vertical asymptote yields $n \approx \frac{2}{\log t}$. However, the intersection of t^n with the horizontal asymptote yields $t^n \approx 1/\log t$ or $n \approx \frac{1}{\log t} \log \frac{1}{\log t}$.

Proof: From Theorem 3, $N^* = \max\{\lceil n \rceil, 1\}$, where n is the solution of the equation

$$t^n = \frac{n-2}{n \log t - 2} \,. \tag{11}$$

It is easy to check that if $\log t \leq 1$, then (cf. Fig. 3)

$$\frac{n-2}{n \log t - 2} \begin{cases} \leq 1 \,; & \text{if } n < \frac{2}{\log t} \,, \\ \geq \frac{1}{\log t} \,; & \text{if } n \geq \frac{2}{\log t} \,. \end{cases}$$

Since $t^n \geq 1$, it follows that the solution of (11) must satisfy $n \geq \frac{2}{\log t}$ or

$$n \log t \geq 2 \,. \tag{12}$$

Now (11) can be written as

$$\begin{aligned} n \log t &= \log \frac{n-2}{n \log t - 2} = \log \frac{1}{\log t} + \log \frac{n \log t - 2 \log t}{n \log t - 2} \\ &\geq \log \frac{1}{\log t} \,, \end{aligned}$$

where we have used (12) and the fact that $\log t \leq 1$ to justify the last inequality. Combining our results we obtain

$$n \log t \geq \max\left\{2, \log \frac{1}{\log t}\right\} .$$

But (12) implies that $n \geq 2$. Hence

$$N^* = \lceil n \rceil \geq \left\lceil \frac{\max\{2, \log \frac{1}{\log t}\}}{\log t} \right\rceil . \qquad \square$$

Proposition 2: If $\log t > 1$, then N^* given in Theorem 3 can be lower bounded by

$$N^* \geq \left\lceil \frac{1.3}{\log t} \right\rceil .$$

Proof: In this case

$$\frac{n-2}{n \log t - 2} \begin{cases} \geq 1 ; & \text{if } n < \frac{2}{\log t} , \\ \leq \frac{1}{\log t} ; & \text{if } n \geq \frac{2}{\log t} . \end{cases}$$

Since $t^n \geq 1$ and $\log t > 1$, it follows that $n < \frac{2}{\log t}$. Now we can write (11) as

$$n \log t = \log \frac{n-2}{n \log t - 2} = \log \frac{1}{\log t} + \log \frac{2 \log t - n \log t}{2 - n \log t}$$
$$< \log \frac{2 \log t - n \log t}{2 - n \log t} \leq \frac{2 \log t - 2}{2 - n \log t} ,$$

where we have used the fact that $\log x \leq x - 1$ to justify the last inequality. Noticing that $2 - n \log t > 0$, we can rewrite the above inequality as follows:

$$(n \log t)^2 - 2(n \log t) + 2 \log t - 2 \geq 0$$

or

$$n \log t \geq 1 \pm \sqrt{3 - 2 \log t} .$$

Noticing that $n \log t > n$, implies that the minus sign solution is refused. Hence

$$N^* \geq \lceil n \rceil \geq \left\lceil \frac{1 + \sqrt{3 - 2 \log t}}{\log t} \right\rceil .$$

At this moment two cases may arise.

Case 1: $1 < \log t \leq 1.3$

Here $1 + \sqrt{3 - 2 \log t} \geq 1.6$. Thus

$$N^* \geq \left\lceil \frac{1.6}{\log t} \right\rceil \geq \left\lceil \frac{1.3}{\log t} \right\rceil .$$

Case 2: $\log t > 1.3$

In this case, we have

$$N^* = \max\{\lceil n \rceil, 1\} \geq 1 \geq \left\lceil \frac{1.3}{\log t} \right\rceil . \qquad \square$$

Using the estimates on N^* given above, we have the following immediate lower bound on R_N^*.

Theorem 4: In an uncoded optical direct-detection OPPM channel with overlapping index $N \in \{1, 2, \cdots\}$ the maximum achievable throughput defined in (8) can be lower bounded as

$$R_N^* \geq R(N, \lceil t^N \rceil) ,$$

where $N = \lceil \frac{a}{\log t} \rceil$, and

$$a \stackrel{def}{=} \begin{cases} \max\{2, \log \frac{1}{\log t}\} ; & \text{if } \log t \leq 1 , \\ 1.3 ; & \text{else.} \end{cases}$$

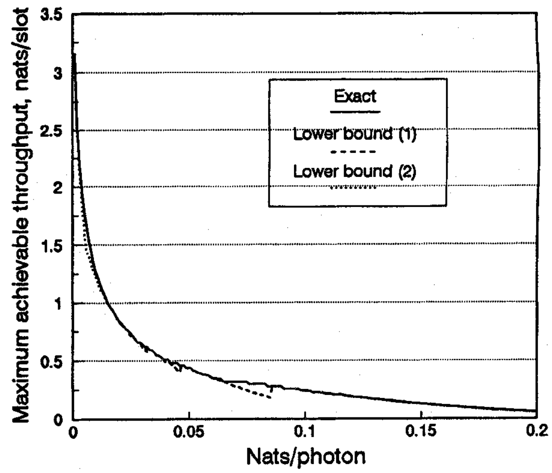

Fig. 4. Maximum achievable OPPM throughput (subject to $P_E \leq 10^{-9}$) versus the information efficiency in nats/photon.

Proof: Immediate from Theorem 3, and Propositions 1 and 2. $\square$

The lower bounds (1) and (2) given by Theorems 3 and 4, respectively, are plotted in Fig. 4 along with the exact solution versus the information efficiency ρ for $\epsilon = 10^{-9}$. The tightness of the bounds are obvious from the figure. It is hard to distinguish between the first lower bound and the exact analysis except for a small range of ρ. The second lower bound (2) coincides with the exact curve especially for the extreme values of ρ.

IV. Multipulse Pulse-PositionModulation

A. MPPM Channel Model

The optical direct-detection MPPM channel model can be found in [7]. In MPPM, P optical pulses, $P \in \{1, 2, \cdots\}$, are transmitted within a time frame (of width T) which is divided into M disjoint slots. Each laser pulse is signaled within one of these slots and thus has a duration $\tau = T/M$. The information is conveyed by the positions of the optical pulses per frame. Since there are $\binom{M}{P}$ pulse patterns, the throughput of MPPM (in nats/slot) is given by

$$R(P, M) \stackrel{def}{=} \frac{\log \binom{M}{P}}{M} . \tag{13}$$

B. MPPM Maximum Throughput

The maximum throughput for every P is given by

$$R_{\max}(P) \stackrel{def}{=} \max_M R(P, M) .$$

As we did in OPPM we would like to find an estimate of the optimum value of the pulse-position multiplicity M^* that gives a maximum throughput. Define

$$f(M) \stackrel{def}{=} \sum_{i=0}^{P-1} \frac{M}{M-i} - \log \binom{M}{P} .$$

It is easy to check that M^* is the solution of the equation $f(M) = 0$. The following lemma gives an estimate of M^*.

Lemma 2: The optimum pulse-position multiplicity of the uncoded optical direct-detection MPPM channel with P optical pulses per frame, $P \in \{1, 2, \cdots\}$, is lower bounded by

$$M^* > 2P \; .$$

Proof: It is easy to check that $f(M)$ is a decreasing function of M. Thus it suffices to show that $f(2P) > 0$. We use the induction method: First, the assertion is true for $P = 1$. Indeed $f(2) = 1 - \log 2 > 0$. Second, assume that it is true for $P = l$, i.e.,

$$f(2l) = \sum_{i=0}^{l-1} \frac{2l}{2l-i} - \log \binom{2l}{l} = \sum_{j=l+1}^{2l} \frac{2l}{j} - \log \binom{2l}{l} > 0 \; .$$

Third, we will show that it should be true for $P = l + 1$. Indeed we can write

$$\begin{aligned} f(2(l+1)) &= \sum_{j=l+2}^{2l+2} \frac{2l+2}{j} - \log \binom{2l+2}{l+1} \\ &= f(2l) + \frac{1}{2l+1} + \sum_{j=l+1}^{2l} \frac{2}{j} - \log \frac{2(2l+1)}{l+1} \\ &> \frac{1}{2l+1} + 2 \log \frac{2l+1}{l+1} - \log \frac{2(2l+1)}{l+1} \\ &= \frac{1}{2l+1} + \log \frac{2l+1}{2(l+1)} \geq \frac{1}{2l+1} + 1 - \frac{2l+2}{2l+1} = 0 \; , \end{aligned}$$

where we have used the assumption of the induction and Appendix C to justify the first inequality, and the fact that $\log y \geq 1 - 1/y$ to justify the last inequality. The induction method is thus complete. □

A good lower bound on the maximum MPPM throughput follows as an immediate result of the above lemma:

$$R_{\max}(P) \geq R(P, 2P + 1) \; .$$

This bound is plotted in Fig. 5 along with the exact values of maximum throughput. The tightness of this bound is obvious from the figure; it is very hard to distinguish between the exact and the lower bound values of $R_{\max}(P)$.

C. Maximum Achievable Throughput for MPPM

The error probability for MPPM is given by [7]

$$P_E(P) = 1 - (1 - e^{-Q})^P \; ,$$

where Q is the photon count per pulse and is related to the information efficiency by

$$Q \overset{def}{=} \frac{\log \binom{M}{P}}{\rho P} \; .$$

The error rate can be upper bounded as $P_E(P) \leq P e^{-Q}$. Thus the maximum throughput that can be achieved so as

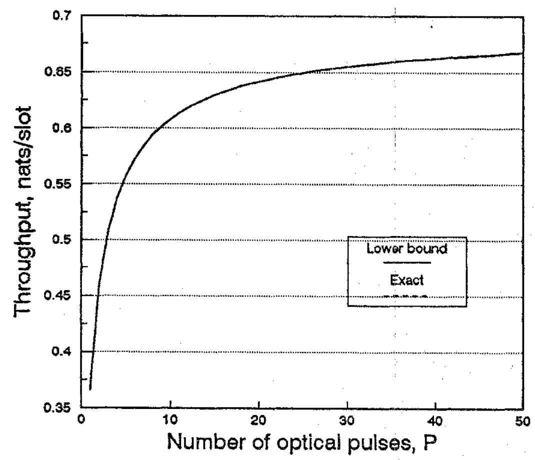

Fig. 5. Maximum MPPM throughput (exact and lower bound) versus the number of optical pulses.

$P_E(P) \leq \epsilon$ is given by

$$R_P^* \overset{def}{=} \max_{\substack{M,P: \\ \binom{M}{P} \geq (\frac{P}{\epsilon})^{\rho P}}} R(P, M) \; .$$

We provide a lower bound on this throughput by noticing that

$$\binom{M}{P} \geq \frac{(M - P + 1)^P}{P!}$$

and making use of Lemma 2. Thus

$$R_P^* \geq \begin{cases} \max_P R(P, 2P+1) \; ; & \text{if } P + 2 \geq (\frac{P}{\epsilon})^{\rho} (P!)^{\frac{1}{P}} \; , \\ \max_P R\Big(P, \lceil P - 1 + (\frac{P}{\epsilon})^{\rho} (P!)^{\frac{1}{P}} \rceil\Big) \; ; & \text{else.} \end{cases} \tag{14}$$

D. A Comparison between OPPM and MPPM Maximum Achievable Throughputs

The exact maximum achievable throughput for OPPM (R_N^*) is compared to the lower bound on R_P^*, given by (14), under a constraint on the error rate of 10^{-9}. The results are plotted in Fig. 6. It is seen from the figure that MPPM provides a better throughput than OPPM if the information efficiency is greater than a certain threshold (about 0.027 nats/photon). Below this threshold, the throughput of MPPM saturates at $\log 2$. However, the throughput of OPPM can be increased as we wish at the expense of both system complexity and large power consumption (small efficiency). To figure out the system complexity, we provide in Table I the corresponding optimum values of N, P, and M. It is obvious that below the threshold OPPM requires a great deal of both synchronization and system complexity in order to achieve larger improvement in throughput than MPPM.

V. Conclusion

Tight upper and lower bounds on the maximum OPPM throughput have been derived under the assumption of quantum-limited direct-detection channels. Simple lower bounds for the maximum achievable throughput of OPPM

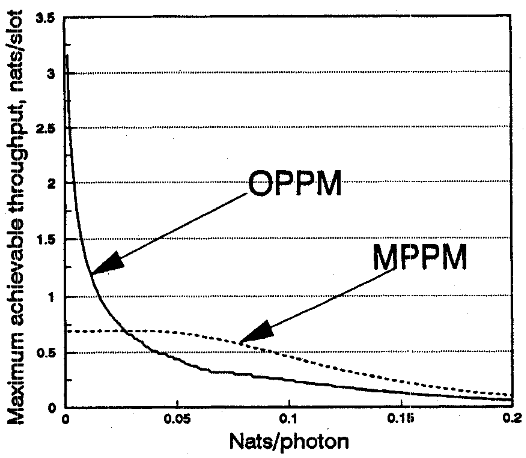

Fig. 6. Maximum achievable throughputs for both OPPM and MPPM (with $P_E \leq 10^{-9}$) versus the information efficiency.

TABLE I
PERFORMANCE AND COMPLEXITY COMPARISONS OF OPPM AND MPPM WITH $\epsilon = 10^{-9}$

ρ, nats/ph	OPPM N	OPPM M	OPPM R	MPPM P	MPPM M	MPPM R
0.001	209.0	88.0	3.161363	1000.0	2001.0	0.691135
0.005	30.0	25.0	1.788264	1000.0	2001.0	0.691135
0.02	5.0	9.0	0.845086	1000.0	2001.0	0.691135
0.04	2.0	6.0	0.511931	1000.0	2115.0	0.689752
0.06	2.0	14.0	0.351874	165.0	456.0	0.647348
0.08	1.0	6.0	0.298627	59.0	224.0	0.564047
0.10	1.0	9.0	0.244136	34.0	186.0	0.461715
0.12	1.0	14.0	0.188504	30.0	247.0	0.359464
0.14	1.0	21.0	0.144977	19.0	236.0	0.270037
0.16	1.0	31.0	0.110774	15.0	287.0	0.197285
0.18	1.0	48.0	0.080650	12.0	356.0	0.141361
0.20	1.0	73.0	0.058773	10.0	462.0	0.099899

(with error probability not exceeding a certain threshold) have been obtained as well. The tightness of these bounds has been examined and compared to the exact values. Lower bounds on the maximum throughput for MPPM have also been included. A comparison between the maximum achievable throughputs for both OPPM and MPPM (under an upper bound on the error probability) has been performed in the last part of the paper. Our results suggest using MPPM for high efficient transmission (exceeding 0.027 nats/photon) and using OPPM for low efficient transmission. The advantage of OPPM (with small ρ) is acquired, however, at the expense of large cost of both system synchronization and complexity.

APPENDIX A

We show that for any two real numbers a and x,

$$\log x \geq a - \frac{e^{a-1}}{x}.$$

Proof: Define the function

$$g(x) \stackrel{def}{=} \log x - a + \frac{e^{a-1}}{x}.$$

It suffices to show that $g(x) \geq 0$. The first derivative of this function is given by

$$g'(x) = \frac{1}{x} - \frac{e^{a-1}}{x^2}.$$

It is easy to check that this function has a global minimum at $x = e^{a-1}$. Hence $g(x) \geq g(e^{a-1}) = 0$. □

APPENDIX B

We show that (in Lemma 1) if $M_N = \sqrt{N} + \frac{N}{\log(N+1)}$, then $g(N) \stackrel{def}{=} f(M_N) \geq 0$. We can see that the first derivative of $g(N)$ is given by

$$g'(N) = \frac{1}{M_N^2} h(N), \tag{B1}$$

where

$$h(N) \stackrel{def}{=} M_N - (N-1)M_N' - M_N M_N'$$

and

$$M_N' = \frac{1}{2\sqrt{N}} + \frac{1}{\log(N+1)} - \frac{N}{(N+1)\log^2(N+1)}.$$

Thus

$$\begin{aligned} h(N) =& \frac{N^2}{(N+1)\log^3(N+1)} - \frac{1}{2} + \frac{N(\sqrt{N}-2)}{(N+1)\log^2(N+1)} \\ &+ \frac{\sqrt{N}}{2} + \frac{1}{2\sqrt{N}} + \frac{1}{\log(N+1)} - \frac{3\sqrt{N}}{2\log(N+1)}. \end{aligned} \tag{B2}$$

The first term is greater than 0.65 and the third term is nonnegative as long as $N \geq 4$. Hence for any $N \geq 4$

$$h(N) > \frac{1}{2\sqrt{N}}\left[1 + N + \frac{2\sqrt{N}}{\log(N+1)} - \frac{3N}{\log(N+1)}\right].$$

From Appendix A we have $\log(N+1) \geq 3 - \frac{e^2}{N+1}$. Thus

$$\begin{aligned} h(N) &> \frac{1}{2\sqrt{N}}\left[1 + N + \frac{2\sqrt{N} - 3N}{3 - \frac{e^2}{N+1}}\right] \\ &= \frac{N+1}{2\sqrt{N}(3N+3-e^2)}\left[2\sqrt{N} - (e^2 - 3)\right]. \end{aligned}$$

The last expression is positive if $N \geq 5$. On the other hand, simple calculations to (B2) lead to $h(N) > 0$ for $N \in \{1, 2, 3, 4\}$. Hence $h(N) > 0$ for any $N \in \{1, 2, \cdots\}$. Substituting in (B1) yields that $g(N)$ is an increasing function in N. Whence $g(N) > g(1) > 0$. □

APPENDIX C

We show that for any two positive integers a and b,

$$\log \frac{b+1}{a} \leq \sum_{i=a}^{b} \frac{1}{i} \leq \log \frac{b}{a-1}.$$

Proof: It is obvious that

$$\int_a^{b+1} \frac{dx}{x} \leq \sum_{i=a}^{b} \frac{1}{i} \leq \int_{a-1}^{b} \frac{dx}{x}.$$

The proof can be completed by performing the two integrations. □

REFERENCES

[1] G. M. Lee and G. W. Schroeder, "Optical PPM with multiple positions per pulse-width," *IEEE Trans. Commun.*, vol. COM-25, pp. 360–364, Mar. 1977.

[2] I. Bar-David and G. Kaplan, "Information rates of photon-limited overlapping pulse position modulation channels," *IEEE Trans. Inform. Theory*, vol. IT-30, pp. 455–464, May 1984.

[3] C. N. Georghiades, "Some implications of TCM for optical direct detection channels," *IEEE Trans. Commun.*, vol. COM-37, pp. 481–487, May 1989.

[4] H. M. H. Shalaby, "Performance of uncoded overlapping PPM under communications constraints," in *Proc. IEEE Int. Conf. Commun. (ICC '93)*, Geneva, Switzerland, May 23-26, 1993, pp. 512–516.

[5] T. Ohtsuki, I. Sasase, and S. Mori, "Overlapping multi-pulse pulse position modulation in optical direct detection channel," in *Proc. IEEE Int. Conf. Commun.*, Geneva, Switzerland, May 23-26, 1993, pp. 1123–1127.

[6] C. N. Georghiades, "Modulation and coding for throughput-efficient optical systems," *IEEE Inform. Theory*, vol. 40, pp. 1313–1326, Sept. 1994.

[7] H. Sugiyama and K. Nosu, "MPPM: A method for improving the band-utilization efficiency in optical PPM," *J. Lightwave Technol.*, vol. 7, pp. 465–472, Mar. 1989.

[8] M. Takahashi, H. Yashima, I. Sasase, and S. Mori, "Capacity and effects of Reed-Solomon codes on multi-pulse PPM in optical communications," *Trans. IEICE Japan*, vol. E72, pp. 1198–1203, Nov. 1989.

[9] T. Ohtsuki, H. Yashima, I. Sasase, and S. Mori, "Cutoff rate and capacity of MPPM in noiseless photon counting channel," *Trans. IEICE Japan*, vol. E74, pp. 4080–4084, Dec. 1991.

Hossam M. H. Shalaby (S'83–M'91) was born in Giza, Egypt, in 1961. He received the B.S. and M.S. degrees from the University of Alexandria, Egypt, in 1983 and 1986, respectively, and the Ph.D. degree from the University of Maryland, College Park, in 1991, all in electrical engineering.

He is currently an Assistant Professor with the Faculty of Engineering, University of Alexandria, Egypt. His research interests include communication theory, optical communications, signal detection and estimation, and information theory.

Dr. Shalaby was a recipient of an SRC fellowship (1987–1991).