Lecture 6

5.1 Example (1):

Consider a photon moving with a velocity c along the x-direction in the S-frame. What id the velocity of the photon as measured in the S`-frame.

Solution

S: $v_x = c \quad , \quad v_y = v_z = 0$

S` : $v`_x = \dfrac{v_x - v}{1 - \dfrac{v_x v}{c^2}} \quad and \quad v_y = v_z = 0$

$$\therefore \; v`_x = \frac{c - v}{1 - \dfrac{cv}{c^2}} = \frac{c - v}{1 - \dfrac{v}{c}} = c\left(\frac{c - v}{c - v}\right) = c$$

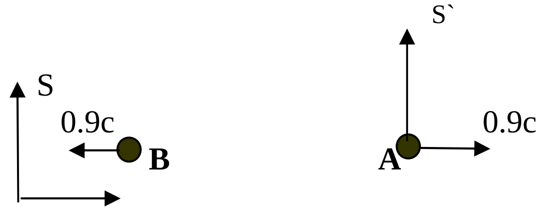
which is in agreement with the fundamental assumption of the special theory of relativity which states that the speed of light is a constant regardless of the relative velocity between the source and the observer.

6.2 Example (2):

Consider two particles A and B traveling at a speed $|v| = 0.9c$ in opposite direction as monitored by the S-frame observer. What would be the relative velocity between the two particles?

Solution

We forst construct a frame S` to move with the particle A, then we find the velocity of B as seen by A, $v`_{xB}$.



$$v`_x = \frac{v_x - v}{1 - \dfrac{v_x v}{c^2}}$$

$$v`_{xB} = \frac{-0.9c - 0.9c}{1 - \dfrac{-0.9c * 0.9c}{c^2}} = \frac{-1.8c}{1 + 0.81} = -0.99c$$

If this problem were to be solved classically the result would be:

$v`_{xB} = -0.9c - 0.9c = -1.8c > c \quad impossible$

6.3 Example(3)

What if these two particles were photons moving with a velocity c.

$$\therefore\ v`_{xB} = \frac{-c-c}{1-\frac{-c*c}{c^2}} = -c$$

6.4 Transformation of acceleration:

We can obtain the equations for acceleration transformation the very same way we have obtained those for velocity transformation.

$$v`_x = \frac{v_x - v}{1-\frac{v_x v}{c^2}}$$

$$dv`_x = \frac{dv_x}{\gamma^2(1-\frac{v_x v}{c^2})^2}$$

$$t` = \gamma(t - \frac{vx}{c^2})$$

$$dt` = \gamma dt(1-\frac{vv_x}{c^2})$$

$$\therefore a`_x = \frac{dv`_x}{dt`} = \frac{a_x}{\gamma^3(1-\frac{vv_x}{c^2})^3}$$

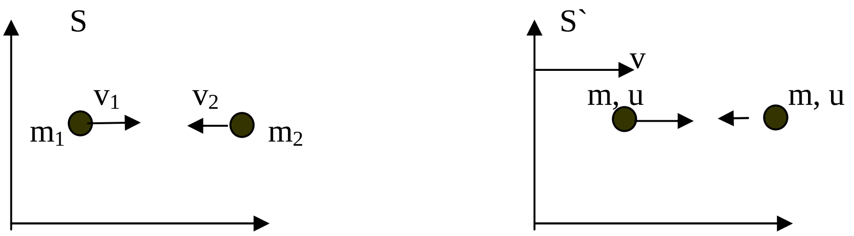
Similarly,

$$a`_y = \frac{1}{\gamma^2(1-\frac{vv_x}{c^2})^2}[a_y + \frac{v_y v}{c^2 - v_x v}a_x]$$

$$a`_z = \frac{1}{\gamma^2(1-\frac{vv_x}{c^2})^2}[a_z + \frac{v_z v}{c^2 - v_x v}a_x]$$

6.5 Relativistic mass:

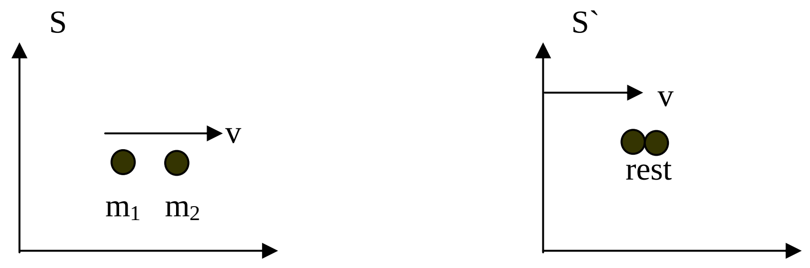
Newton's second law $\Sigma F_i = ma$ should be kept invariant in all inertial reference frames. The acceleration transformation has shown that acceleration is different in different frames. To satisfy the first postulate the mass has also in turn to be different. So, we should now seek a transformation equation for the mass. Consider a case of inelastic collision between two masses m_1 and m_2.



Before Collision

Two particles of the same mass m moving in opposite direction with the same velocity u are assumed in the S`-frame. These particles are seen by O to be of masses

m_1 and m_2 moving oppositely with velocities v_1 and v_2. Consider now an inelastic collision to take place. The collision, inelastically, means that the two particles have been merged and brought to rest as show.



After Collision

Applying now the principle of conservation of momentum in both frames after and before collision:

S : $m_1 v_1 + m_2 v_2 = m_1 v + m_2 v$ (1)

S`: $mu + m(-u) = 0$ (2)

Divide equation (!) by m_2, you get:

$$\frac{m_1}{m_2} v_1 + v_2 = \frac{m_1}{m_2} v + v$$

$$\frac{m_1}{m_2}(v_1 - v) = (v - v_2)$$

$$\therefore \frac{m_1}{m_2} = \frac{v - v_2}{v_1 - v}$$

We also have from the inverse transformation that:

$$v_1 = \frac{u + v}{1 + \frac{uv}{c^2}}$$

$$v_2 = \frac{-u + v}{1 - \frac{uv}{c^2}}$$

$$\therefore \frac{m_1}{m_2} = \frac{1 + \frac{uv}{c^2}}{1 - \frac{uv}{c^2}}$$

We can also show that:

$$\frac{1 + \frac{uv}{c^2}}{1 - \frac{uv}{c^2}} = \frac{\sqrt{1 - \frac{v_2^2}{c^2}}}{\sqrt{1 - \frac{v_1^2}{c^2}}}$$

$$\therefore \ \frac{m_1}{m_2} = \frac{\sqrt{1-\frac{v_2^2}{c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}}$$

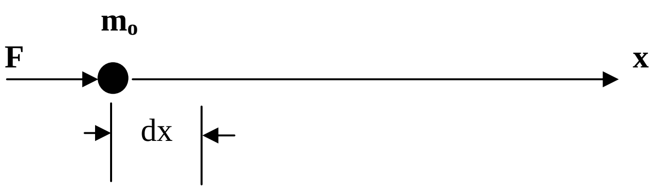
Consider now the case where the particle m_2 to be at rest, i.e. $v_2 = 0$, and denote its mass by m_o, "Rest masses are invariant in all inertial frames".

$$\boxed{m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_o}$$

i.e. masses increase as their velocities increase.

6.6 Kinetic Energy of a Relativistic mass:

Suppose now a force F is exerted on a particle whose rest mass is m_o and the particle has been displaced a distance dx along the x-axis. The particle is said to have gained an amount of kinetic energy equal to the work done by the force.



$$K.E. = \int_{rest}^{final} F.dx$$

$$F = \frac{dmv}{dt}$$

$$K.E. = \int_{rest}^{final} d(mv)\frac{dx}{dt}$$

$$= \int_{rest}^{final} v.(mdv + vdm)$$

Applying the relativity of mass we reached at previously:

$$m = \gamma m_o$$

$$m^2 = \gamma^2 m_o^2$$

$$m^2(1-\frac{v^2}{c^2}) = m_o^2$$

$$m^2c^2 - m^2v^2 = m_o^2c^2$$

Differentiating both sides yields:

$$2mc^2\,dm - (2mv^2\,dm + 2m^2\,vdv) = 0$$

$$mc^2\,dm = v(mvdm + m^2\,dv)$$

$$c^2\,dm = v(vdm + mdv)$$

$$\therefore K.E. = \int_{rest}^{final} c^2 dm$$

$$\boxed{K.E. = mc^2 - m_o c^2}$$

i.e. the kinetic energy is the difference between the total energy mc^2 and the rest mass energy $m_o c^2$.

$$E = mc^2 = K.E. + m_o c^2$$

So, a mass m has a total energy mc^2, i. e. the conversion factor between mass and energy is c^2.
Again we can check the correspondence principle:
If $v \ll c$

$$\therefore K.E. = mc^2 - m_o c^2$$
$$= m_o c^2 (\gamma - 1)$$
$$= m_o c^2 [(1 - \frac{v^2}{c^2})^{-\frac{1}{2}} - 1]$$
$$\cong m_o c^2 [1 + \frac{1}{2}\frac{v^2}{c^2} - .... - 1]$$
$$= \frac{1}{2} m_o v^2 \equiv classical \ \ value$$

6.7 Energy, Momentum and Kinetic energy

A useful relation can be derived relating the total energy to momentum and form a right-angled triangular as follows:

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

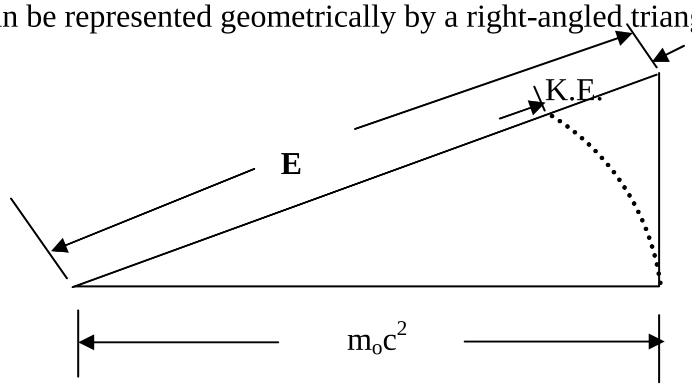
$$m^2 (1 - \frac{v^2}{c^2}) = m_o^2 \ \times c^4$$

$$m^2 c^4 - m^2 v^2 c^2 = m_o^2 c^4$$

$$E^2 - p^2 c^2 = m_o^2 c^4$$

Or $E^2 = p^2 c^2 + m_o^2 c^4$

Which can be represented geometrically by a right-angled triangle.



End of Chapter 1