Control Systems And Their Components (EE391)

Lec. 10: Closed loop SS Control (Reference Inputs and State Estimators)

Thu. April 28th, 2016

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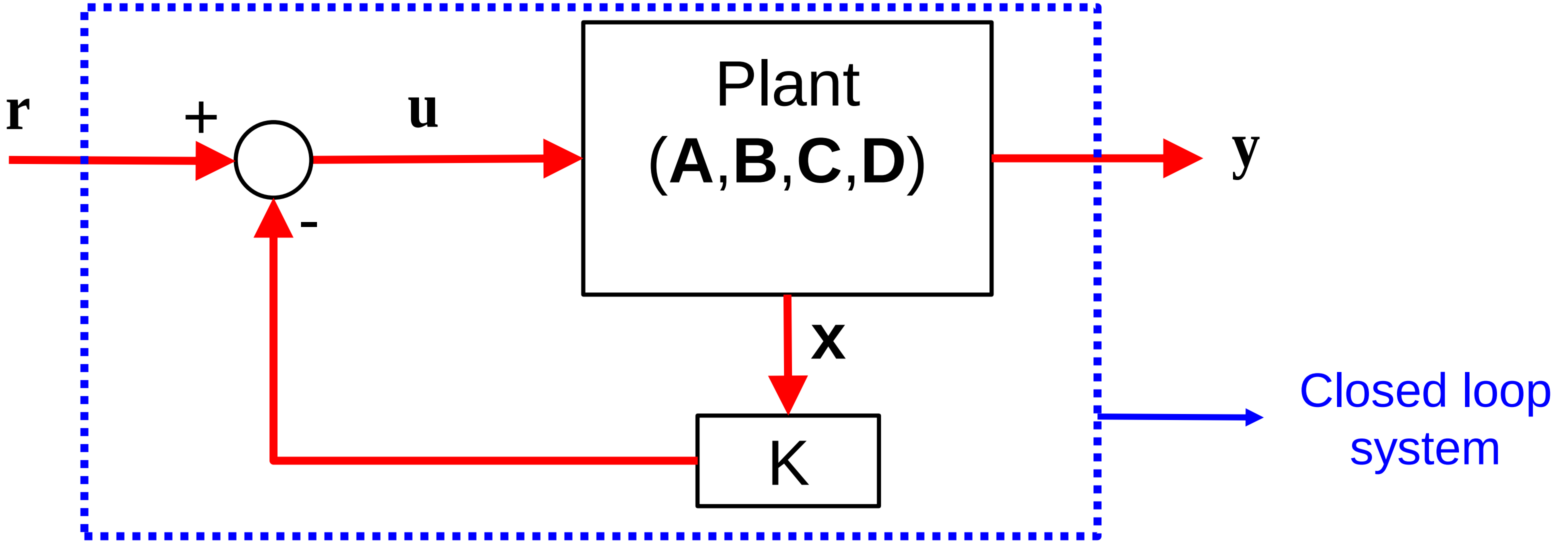
Lecture Outline

- Adding reference inputs to the design of state feedback controllers
- Pre-scaling reference inputs to achieve zero steady state error
- Introduction to state estimators/observers

Full state feedback (from last lecture)



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Assume full state feedback of the form

$$\mathbf{u}(t) = \mathbf{r}(t) - \mathbf{K}\mathbf{x}(t)$$

where **r** is a reference input and $\mathbf{K} \in \mathbf{R}^{1\times n}$ (assume a single input for simplicity)

Full state feedback with reference input

$$\dot{\mathbf{x}}(t) = \underbrace{\left(\mathbf{A} - \mathbf{BK}\right)}_{\mathbf{A}_{cl}}\mathbf{x}(t) + \mathbf{Br}(t)$$

Now we will not assume **r** = 0 as we did in regulator case, and assume we have a certain reference input **r** ≠ 0 that we would like the output **y** to track (what does this mean?)

$$y(t) = \mathbf{Cx}(t), \quad \text{assume } \mathbf{D} = 0$$

A sufficient condition on the output **y** for tracking **r** is

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} r(t)$$

assuming a single input for simplicity

which makes $e_{ss} = 0$ (zero steady state error)

Full state feedback with reference input

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} r(t)$$

Final value theorem

$$\lim_{s \to 0} sY(s) = \lim_{s \to 0} sR(s)$$

$$\lim_{s \to 0} \frac{Y(s)}{R(s)} = 1$$

Closed loop transfer function at DC equals 1

Full state feedback with reference input

Illustrative Example

Find the feedback gains **K** of the following SS system such that the closed loop poles become -2+2i and -2-2i, then find the closed loop TF at DC

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

Solution

From last lecture $\mathbf{K} = \begin{bmatrix} 7 & 3 \end{bmatrix}$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}(t) + \mathbf{B}r(t)$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

Full state feedback with reference input

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$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

Solution

From last lecture $\mathbf{K} = \begin{bmatrix} 7 & 3 \end{bmatrix}$

$$\therefore TF = \frac{Y(s)}{R(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A_{cl}}\right)^{-1}\mathbf{B} = \mathbf{C}\left(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right)^{-1}\mathbf{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 8 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 4s + 8} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\frac{1}{s^2 + 4s + 8}}$$

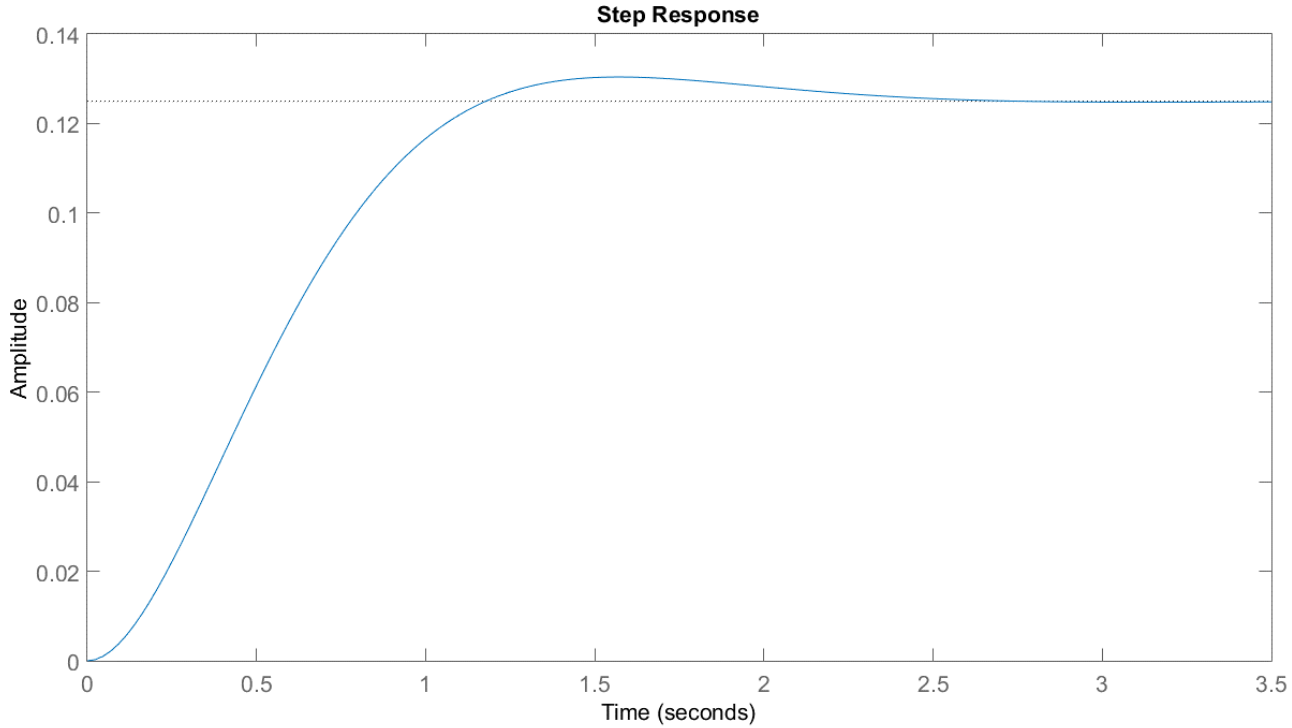
Full state feedback with reference input

Illustrative Example

$$\therefore \lim_{s \to 0} \frac{Y(s)}{R(s)} = \frac{1}{8} \neq 1$$

```
sys = ss(A-B*K,B,C,D)
step(ss)
```

$y_{ss} = 0.125$
$r_{ss} = 1$



How can we overcome this lack of tracking?

Pre-scaling reference input

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

Solution

If we pre-scale the reference input r by a factor $\overline{N}$ before entering closed loop system

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}(t) + \mathbf{B}\overline{N}\,r(t)$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overline{N}\,r(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

Pre-scaling reference input

Illustrative Example

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}\mathbf{u}(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(t)$$

Solution

If we pre-scale the reference input r by a factor $\overline{N}$ before entering closed loop system

$$\therefore TF=\frac{Y(s)}{R(s)}=\mathbf{C}\left(s\mathbf{I}-\mathbf{A_{cl}}\right)^{-1}\overline{N}\,\mathbf{B}=\mathbf{C}\left(s\mathbf{I}-\left(\mathbf{A}-\mathbf{BK}\right)\right)^{-1}\overline{N}\,\mathbf{B}$$

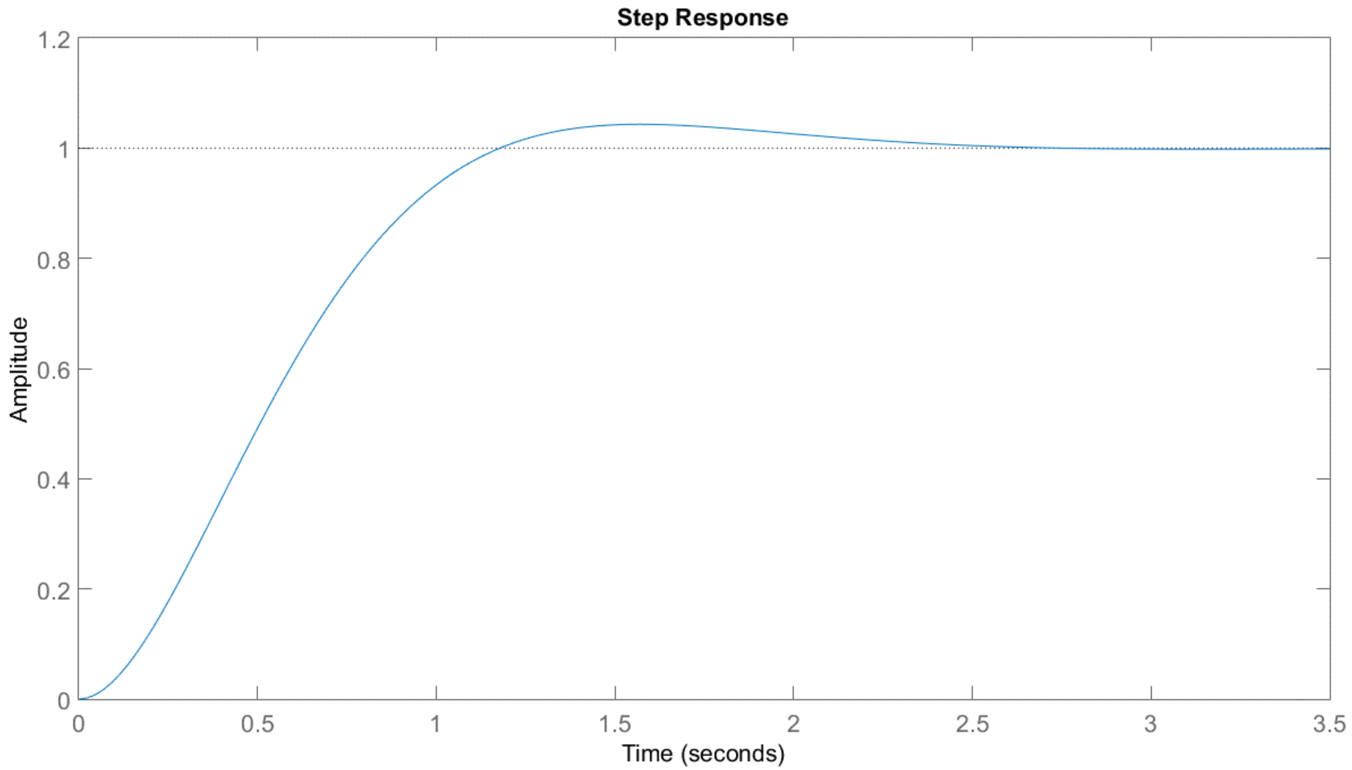
$$=\frac{\overline{N}}{s^{2}+4s+8}$$

$$\therefore \lim_{s\to 0}\frac{Y(s)}{R(s)}=\frac{\overline{N}}{8}\quad\Rightarrow\quad \boxed{\overline{N}=8}$$

Pre-scaling reference input

Illustrative Example

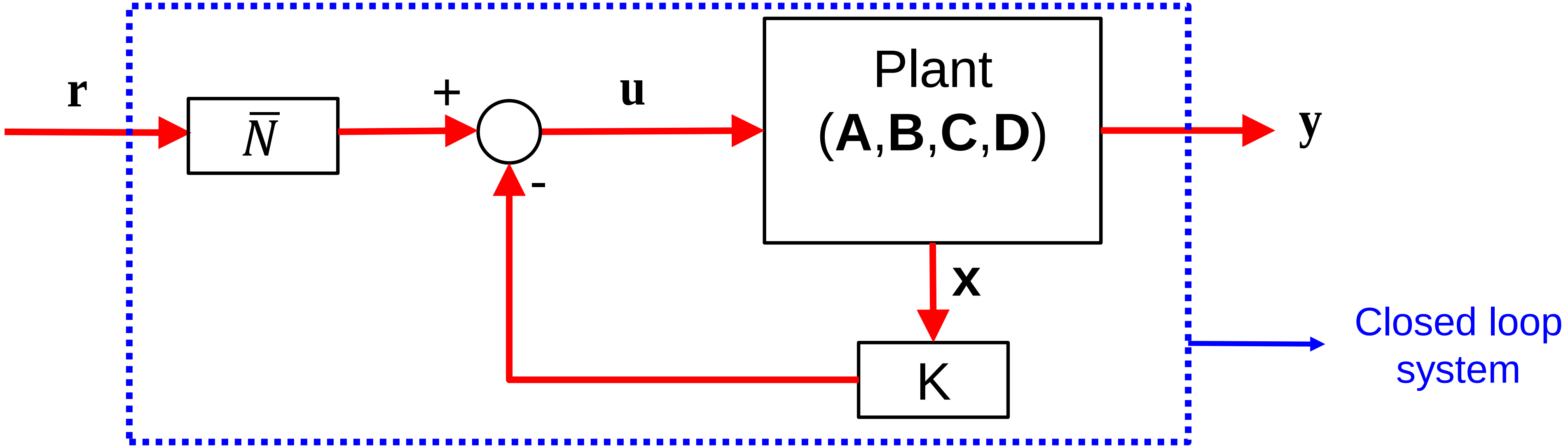
```
sys = ss(A-B*K,Nbar*B,C,D)
step(ss)
```



$y_{ss} = 1$
$r_{ss} = 1$

Problem solved with pre-scaling *r*

Full state feedback with pre-scaled reference (summary)



Assuming full state feedback and pre-scaled reference input to achieve Steady state tracking

$$u(t) = \bar{N}r(t) - \mathbf{K}\mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\bar{N}r(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

$$\bar{N} = \frac{1}{\mathbf{C}(-\mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B}}$$

MATLAB example 1 (2^{nd} order system)

$$G(s) = \frac{8}{(s-5)(s-10)}$$

$$\text{Maximum overshoot} = 5\%$$

$$2\% \text{ settling time} = 4 \text{ sec}$$

- Find the desired two poles
- Find K that achieves so (same as last lecture)
- Find Nbar that achieves steady state tracking
- Make sure closed loop system satisfy the specifications

MATLAB example 2 (3^{rd} order system)

$$G(s) = \frac{8}{(s-5)(s-10)(s-7)}$$

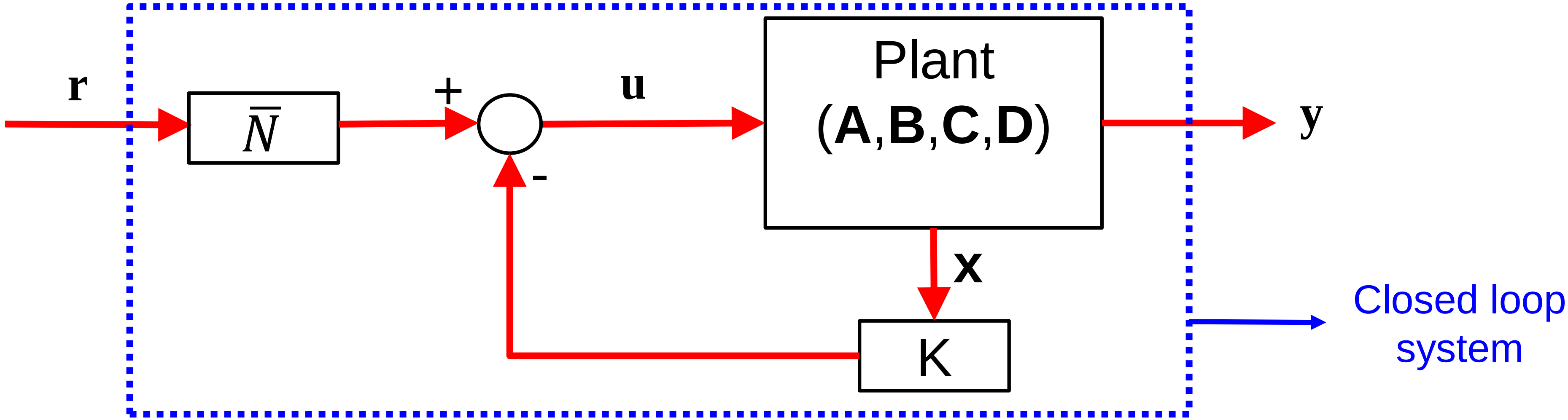
$$\text{Maximum overshoot} = 5\%$$

$$2\% \text{ settling time} = 4 \text{ sec}$$

- Find the desired two poles
- Place the remaining pole far away
- Find K that achieves so (same as previous lecture)
- Find Nbar that achieves steady state tracking
- Make sure closed loop system satisfy the specifications

Intro to State estimators / observers

- What we did so far



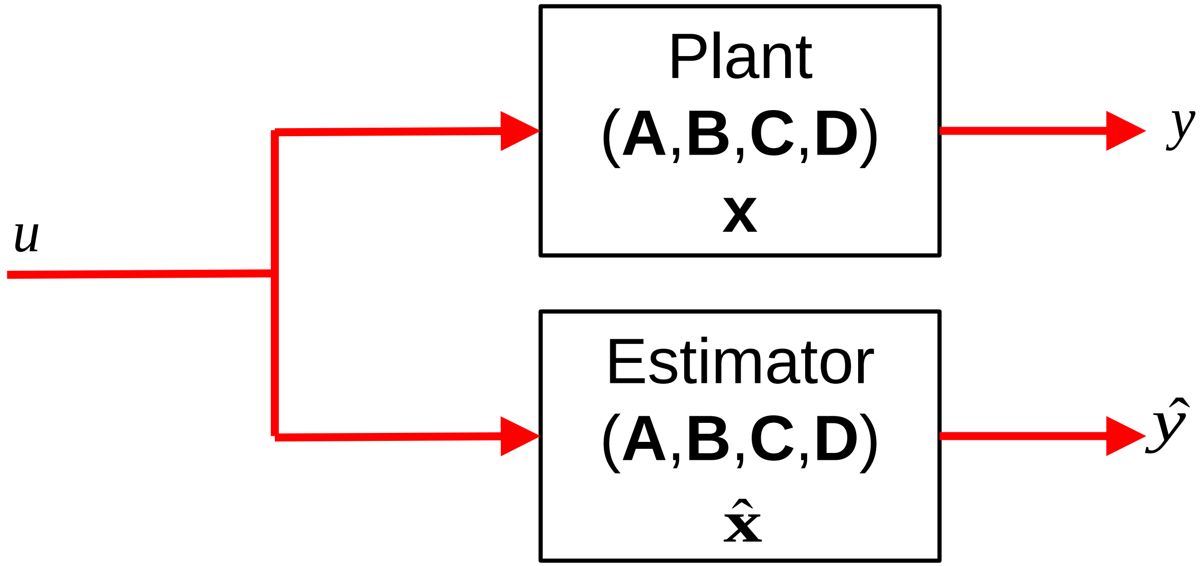
- Problem is that we have assumed **full state feedback** which means we have full access to the state variables of the system from which $u = \bar{N} r - \mathbf{K}\mathbf{x}$ is evaluated
- This is not true since in reality we only have access to the sensor outputs y and not the state variables **x**
- Could try output feedback but will have less degrees of freedom compared to state feedback (cannot control all pole locations freely like what we did with **K**)

Intro to State estimators / observers

- The solution to the lack of measurements of **x** is to use **state estimator/observer**
- A state estimator/observer is a replica of the actual system or plant that tries to estimate the true state variables of the system from the actual measured output y and provides the estimated state vector $\hat{\mathbf{x}}$
- We can then combine the developed estimator together with state feedback control to have a realistic method of controlling the closed loop poles based on the feedback of estimated state variables $u = -\mathbf{K}\hat{\mathbf{x}}$ (more on this later but we will focus on estimator alone for the moment)

- Estimation strategies we have in hand
  - Open loop (bad strategy as we will see)
  - Closed loop

Open loop estimator



- Assuming we know the input *u* and plant matrixes A, B, C, and that D = 0
- We can just simulate a replica of the actual plant on say a computer and obtain an estimate $\hat{\mathbf{x}}$ as follows

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t)$$

Open loop estimator

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$ ⇦ Dynamic eq. of actual plant

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t)$$ ⇦ Dynamic eq. of simulated plant (estimator)

$$\text{If} \quad \mathbf{x}(0) = \hat{\mathbf{x}}(0), \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \forall t$$

- However we do not know **x**(0) so how well the above estimator works if the initial estimation error is not zero
- Define the estimation error **e**(t)

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$$

$$\frac{d}{dt}\left\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right\} = \mathbf{A}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t)$$

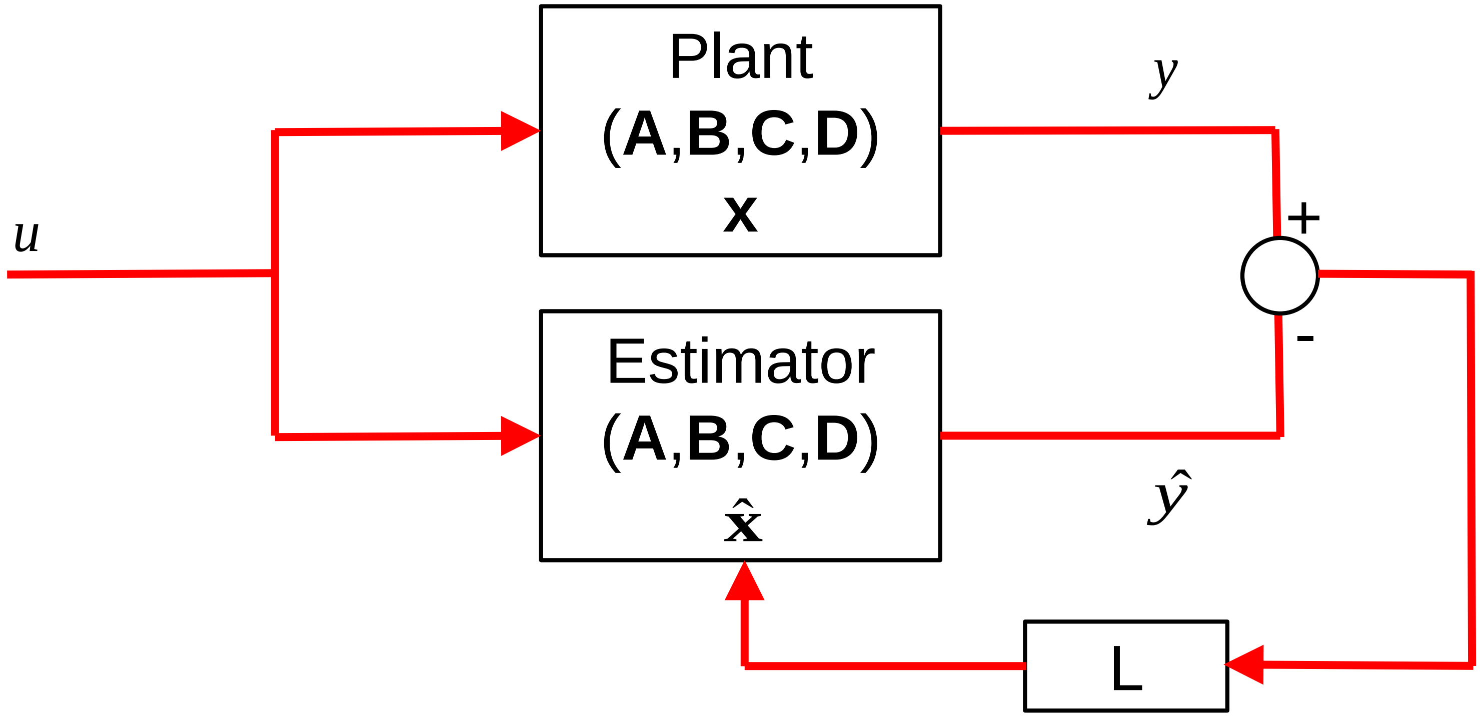
$$\therefore \boxed{\mathbf{e}(t) = e^{\mathbf{A}t}\mathbf{e}(0)}$$

Open loop estimator

$$\mathbf{e}(t) = e^{\mathbf{A}t}\mathbf{e}(0)$$

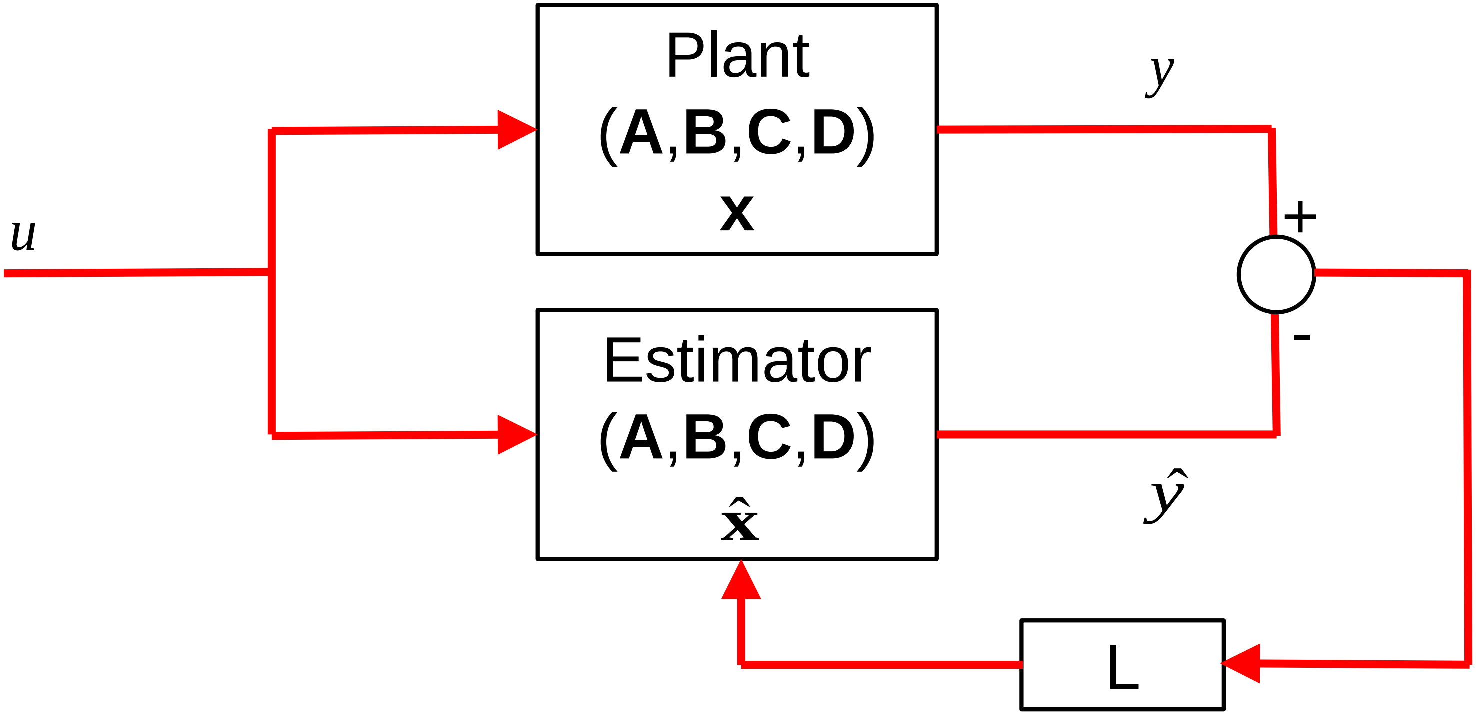
- Everything looks fine if initial error **e**(0) = 0
- If **e**(0) ≠ 0, **e**(t) as $t \to \infty$ may decay to zero if the eigenvalues of **A** have negative real part (if the original plant is stable)
- Since the estimation error is totally dependent on **A**, this is not a good estimation strategy since we cannot control the dynamics of the estimation error at all
- We may make use of other available information in building a better state estimator (how? → closed loop estimator)

Closed loop estimator



- The idea is to feedback the error in the estimated output, i.e. its difference from the actual output of the system which can be observed
- **L** is a selectable gain matrix (similar to **K**) that will allow us to control the dynamics of the estimation error **e**(t) as will be seen

Closed loop estimator



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

⇦ Dynamic eq. of actual plant

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$

⇦ Dynamic eq. of simulated plant (estimator)

Closed loop estimator

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$ ⇦ Dynamic eq. of actual plant

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$ ⇦ Dynamic eq. of simulated plant (estimator)

$$y(t) = \mathbf{C}\mathbf{x}(t)$$ ⇦ Output eq. of actual plant

$$\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$$ ⇦ Output eq. ofestimator

- Let's try to find the dynamics of **e**(t) with the added feedback to the estimator

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) - \mathbf{L}\left(y(t) - \hat{y}(t)\right)$$

$$= \mathbf{A}\mathbf{e}(t) - \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

$$= \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e}(t)$$

Closed loop estimator

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{LC}\right)\mathbf{e}(t)$$

$$\therefore \boxed{\mathbf{e}(t) = e^{\left(\mathbf{A}-\mathbf{LC}\right)t}\,\mathbf{e}(0)}$$

- It is obvious that by choosing a proper gain matrix **L**, we can control the dynamics of the estimation error, i.e. make it go to zero fast such that the estimated state variables converge to the actual state variables fast enough
- This is all controlled by the eigenvalues of **A-LC**

$$\boxed{\left|s\mathbf{I} - \left(\mathbf{A} - \mathbf{LC}\right)\right| = \prod_{j=1}^{n}\left(s - s_j\right) = 0}$$

Desired pole locations of state estimator
where? → we will see later

Controller and Observer design (Dual problems)

Controller design

$$\mathbf{K} \in \mathbf{R}^{1 \times n}$$

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{BK}) \right| = \prod_{j=1}^{n} (s - s_j) = 0$$

Desired pole locations closed loop sys

Observer/Estimator design

$$\mathbf{L} \in \mathbf{R}^{n \times 1}$$

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{LC}) \right| = \prod_{m=1}^{n} (s - s_m) = 0$$

Desired pole locations of state estimator

- **K** and **L** are chosen to achieve desired pole locations
- **Controller** and **Observer** design are called **dual problems**
- Just like before when the system had to be **controllable** to find **K**, the system now has to be **observable** to find **L**

Akermann’s formula for Observer design

- It gives a formal way to obtain **L**
- Without proof

$$\mathbf{L} = \mathbf{O}_n^{-1}\boldsymbol{\Phi}_{desired}\left(\mathbf{A}\right)\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}^{\mathbf{T}}$$

Observability matrix

Desired characteristic equation of state estimator/observer

- Clearly $\mathbf{O}_n$ needs to be invertible, hence full rank, hence the system must be **observable** in order to be able to find **L**

Observer design

Example For the system with the following matrices and initial state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Test observability
- Find **L** that makes poles of estimator/observer at -3 and -4

Observer design

Solution

For the system with the following matrices and initial state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Test observability

$$\mathbf{O}_n = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\{\mathbf{O}_n\} = 2 \quad \Rightarrow \quad \text{observable}$$

Observer design

Solution

For the system with the following matrices and initial state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Find **L** that makes poles of estimator/observer at -3 and -4

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{LC}) \right| = \prod_{m=1}^{n} (s - s_m) = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right| = (s+3)(s+4)$$

$$\left| \begin{bmatrix} s+1+L_1 & -1.5 \\ L_2 - 1 & s+2 \end{bmatrix} \right| = (s+3)(s+4)$$

Observer design

Solution

For the system with the following matrices and initial state vector

$$\mathbf{A} = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

- Find **L** that makes poles of estimator/observer at -3 and -4

$$\left| \begin{bmatrix} s+1+L_1 & -1.5 \\ L_2-1 & s+2 \end{bmatrix} \right| = (s+3)(s+4)$$

$$s^2 + (L_1+3)s + (2L_1 + 1.5L_2 + 0.5) = s^2 + 7s + 12$$

$$L_1 = 4$$

$$L_2 = 2.333$$

MATLAB

Use $\mathbf{A}^T$ and $\mathbf{C}^T$ as your **A** and **B** in "place" function

L = place(A.',C.',desired poles)