Control Systems And Their Components (EE391)

Lec. 4: State Space Representation and its Dynamic Solution

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Lecture Outline

- Introduction to State Space (SS) representation
- State Space Equations
- Dynamic response of SS equations (Transient Solution)

Introduction to State Space Representation

Classical Control

- SISO
  (Single Input Single Output)
- Low order ODEs
- Time-invariant
- Transfer function based approaches (Root-Locus and Frequency domain design approaches)
- Continuous, analog
- Before 80s

Modern Control

- MIMO
  (Multiple Input Multiple Output)
- High order ODEs -> System of linear 1^{st} order DEs
- Time-invariant and time variant
- State space approach
- Discrete, digital
- 80s and after

- The difference between classical control and modern control originates from the different modeling approach used by each control
- The modeling approach used by modern control enables it to have new features and ability to control much more complicated systems compared to classical control

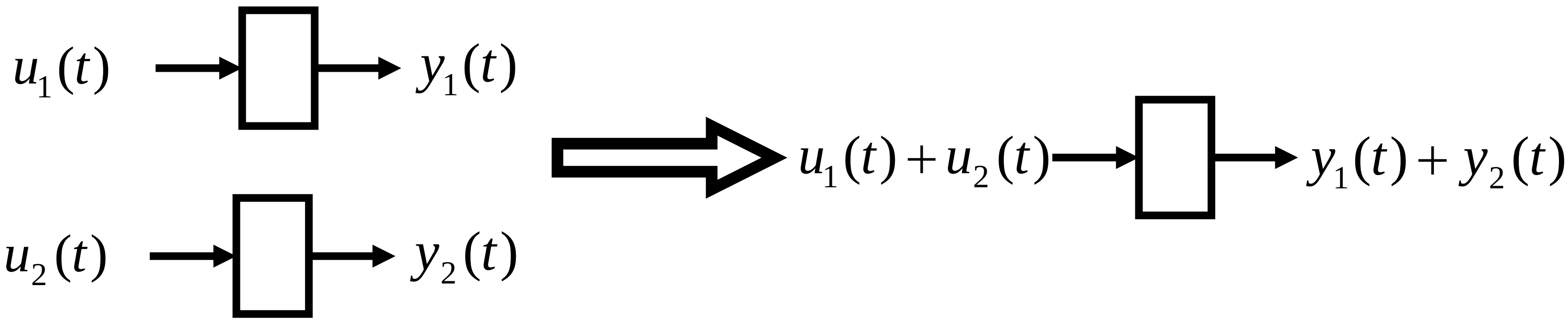
Refresher of System Classifications

- Systems are classified based on:
  - *The number of inputs and outputs*: single-input single-output (SISO), multi-input multi-output (MIMO), MISO, SIMO
  - *Existence of memory*: if the present output depends on the present input only, then the system is said to be memoryless (or static), otherwise it has memory (e.g. purely resistive circuit)
  - *Causality*: a system is called causal if the output depends only on the present and past inputs and independent of the future inputs
  - *Dimensionality*: the dimension of system can be finite (lumped) or infinite (distributed)
  - *Linearity*: a linear system satisfies the properties of superposition and homogeneity
  - *Time-Invariance*: a time-invariant (TI) system has static characteristics that do not change with time.

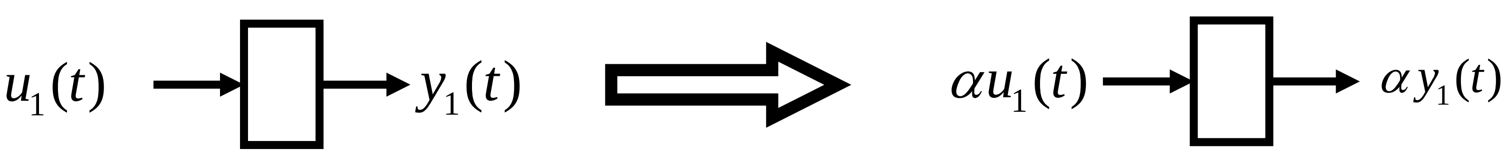
Refresher of System Classifications

Linearity:

- A system is said to be linear if it satisfies the following two properties of superposition and homogeneity
  - Superposition



  - Homogeneity



Refresher of System Classifications

- Linearity:

Example: Check the linearity of the following system (governed by ODE).

$$u(t) \longrightarrow \boxed{y''(t)+2y'(t)+y(t)=u'(t)+3u(t)} \longrightarrow y(t)$$

Let $y_1''(t)+2y_1'(t)+y_1(t)=u_1'(t)+3u_1(t)$

$y_2''(t)+2y_2'(t)+y_2(t)=u_2'(t)+3u_2(t)$

Then $[\alpha u_1(t)+\beta u_2(t)]'+3[\alpha u_1(t)+\beta u_2(t)]$

$$=\alpha u_1'(t)+\beta u_2'(t)+\alpha 3u_1(t)+\beta 3u_2(t)$$

$$=\alpha[u_1'(t)+3u_1(t)]+\beta[u_2'(t)+3u_2(t)]$$

$$=\alpha[y_1''(t)+2y_1'(t)+y_1(t)]+\beta[y_2''(t)+2y_2'(t)+y_2(t)]$$

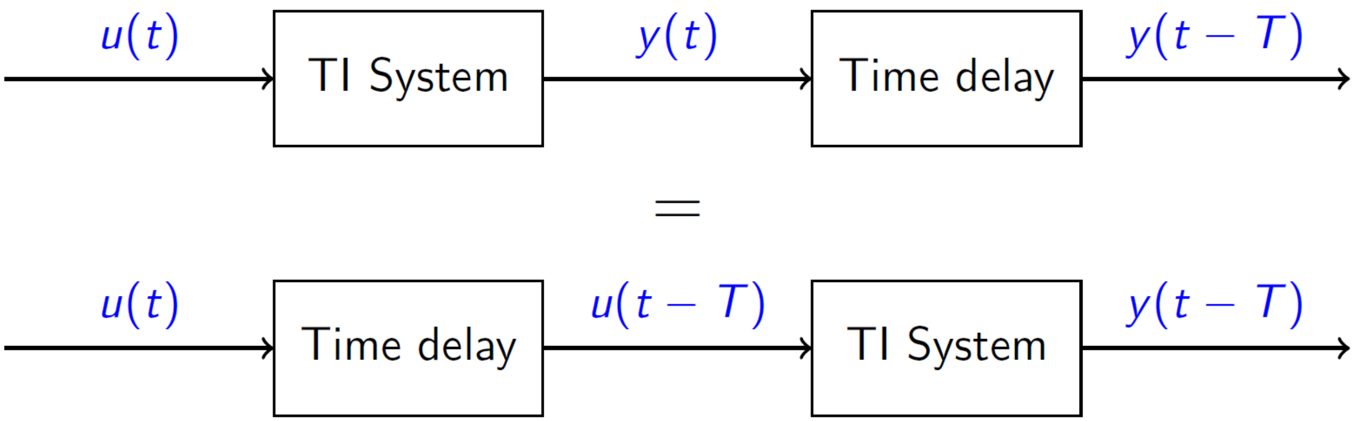
$$=[\alpha y_1(t)+\beta y_2(t)]''+2[\alpha y_1(t)+\beta y_2(t)]'+[\alpha y_1(t)+\beta y_2(t)]$$

→ The system is **linear**

Refresher of System Classifications

- Time Invariance:

- A system is said to be time-invariant if it commutes with time delays. In other words, the output to a time delayed input is the same as the time delayed output to the original input



- A system time-invariant if its parameters do not change over time.

Refresher of System Classifications

Causality

- A system is causal if the output at time t_0 depends only on the input up to and including time t_0
- A causal system has memory
- A causal system has the following condition on its impulse response

$$h(t) = 0 \qquad \forall t < 0$$

State Space Equations

- We know that for a causal system, in order to compute the output at a given time t_0, we need to know the input signal over $(-\infty, t_0] \rightarrow$ "A lot of information!!"
- The central question is: "Is there a more manageable way to express the entire memory or history of the past inputs to the system?"

State Variables

They are the set of variables $x_1(t), x_2(t), x_3(t), \ldots x_n(t)$ such that the knowledge of these variables at time t_1 together with the input u between times t_1 and t_2 is sufficient to uniquely evaluate: a) the output at time t_2, and b) the updated state variables at t_2

- In other words, these state variables is the concise way to summarize the entire history of past inputs to the system, i.e. knowledge of present values of the state variables as well as present and future inputs is sufficient to obtain future output
- The state of the system is usually a vector **x**[t] in an *n*-dimensional space $\mathbb{R}^n$

State Space Equations

- There choice of state variables of a system is not unique, i.e. there are infinite choices (realizations) of the same system
- What is key is the minimum number of state variables needed to fully represent or realize the system which is called “minimal realizations”

Dimension of a system

It is the minimum number of state variables sufficient to describe the system state

- Throughout the course, We deal only with finite dimensional systems
- Mostly systems with lumped elements are finite dimensional whereas systems with distributed component models are infinite dimensional (e.g. transmission lines)

State Space Equations

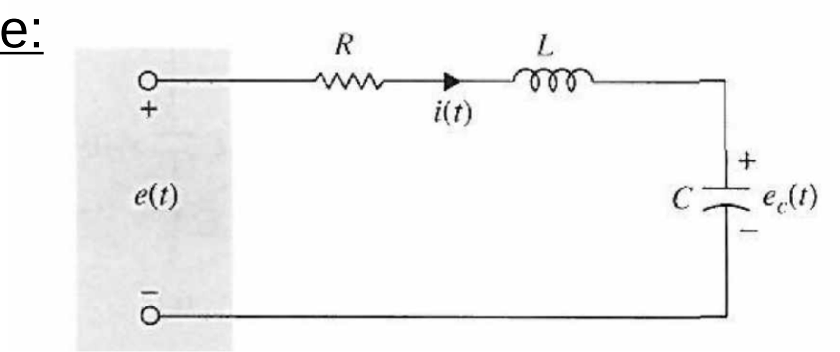
The key idea in state space equations is that you break the n^{th} order Diff eq. that represents a system into n 1^{st} order Diff. equations

- Since the entire SS formulation involve matrices that describe the system behavior, it is naturally suited for multiple-input multiple-output (MIMO) systems compared to transfer function models
- Matrices mean **a lot of Linear Algebra!!!!**

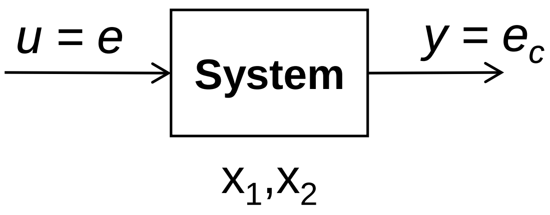
- Let's start with an example showing how an 2^{nd} order Diff. eq. is broken into two 1^{st} order Diff. Eqs giving us the SS model before getting into the general SS formulation

State Space Equations

Example:



$\Leftrightarrow$



$$e(t) = Ri(t) + L\frac{di}{dt} + e_c(t) \quad \Leftarrow \quad i(t) = C\frac{de_c(t)}{dt}$$

$$e(t) = RC\frac{de_c(t)}{dt} + LC\frac{d^2e_c(t)}{dt^2} + e_c(t)$$

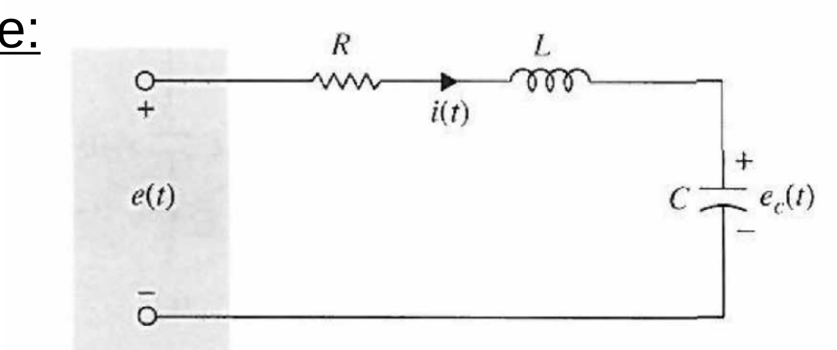
$$\boxed{LC\ddot{e}_c + RC\dot{e}_c + e_c = e}$$

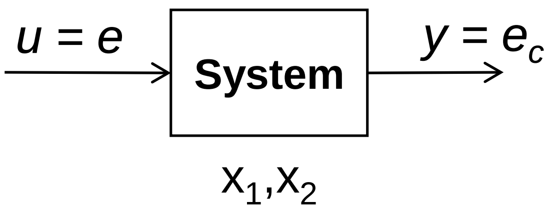
Let us take a possible state variable assignment

$$\boxed{x_1 = e_c \quad and \quad x_2 = \dot{e}_c}$$

State Space Equations

Example:





$$\boxed{LC\ddot{e}_c + RC\dot{e}_c + e_c = e}$$

Let us take a possible state variable assignment

$$\boxed{x_1 = e_c \quad and \quad x_2 = \dot{e}_c}$$

Then $\dot{x}_1 = \dot{e}_c = x_2$

$$\dot{x}_2 = \ddot{e}_c = -\frac{R}{L}\dot{e}_c - \frac{1}{LC}e_c + \frac{1}{LC}e$$

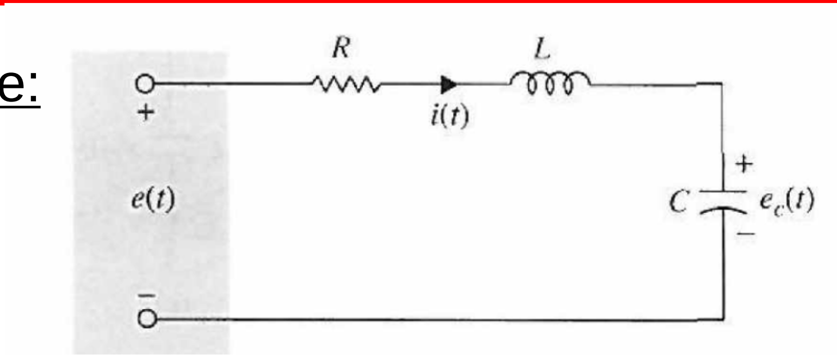
$$= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{LC}u$$

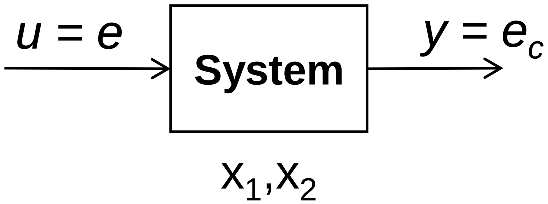
$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{LC}u \end{aligned}}$$

System of two 1st order Diff. Eqs.

State Space Equations

Example:





$$\boxed{x_1 = e_c \quad and \quad x_2 = \dot{e}_c}$$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{R}{L}x_2 - \frac{1}{LC}x_1 + \frac{1}{LC}u \end{aligned}}$$

In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- System of two coupled 1st order Diff. Eqs. describing the dynamic behavior of state variables
- Another equation to calculate the output in terms of the state variables

State Space Equations

- <u>General case</u>: For an n^th order Diff. Eq. of SISO system

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} + \ldots + a_n y(t) = u(t)$$

Using the following state variables assignments (*n* state variables)

$$x_1(t) = y(t), \quad x_2(t) = \frac{dy(t)}{dt}, \quad x_3(t) = \frac{d^2 y(t)}{dt^2}, \quad x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}}$$

Then $\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4$

$$\dot{x}_n = \frac{d^n y(t)}{dt^n}$$

$$= -a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} - a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} - \ldots - a_n y(t) + u(t)$$

State Space Equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4$$

$$\dot{x}_n = \frac{d^n y(t)}{dt^n} = -a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} - a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} - \ldots - a_n y(t) + u(t)$$

$$= -a_1 x_n - a_2 x_{n-1} - \ldots - a_n x_1 + u(t)$$

In Matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + 0 \cdot u(t)$$

State Space Equations

For an *n* dimensional system with *p* inputs and *m* outputs

$$\overset{n\times 1}{\dot{\mathbf{x}}(t)} = \underset{n\times n}{\mathbf{A}}\overset{n\times 1}{\mathbf{x}(t)} + \underset{n\times p}{\mathbf{B}}\overset{p\times 1}{\mathbf{u}(t)}$$

$$\overset{m\times 1}{\mathbf{y}(t)} = \underset{m\times n}{\mathbf{C}}\overset{n\times 1}{\mathbf{x}(t)} + \underset{m\times p}{\mathbf{D}}\overset{p\times 1}{\mathbf{u}(t)}$$

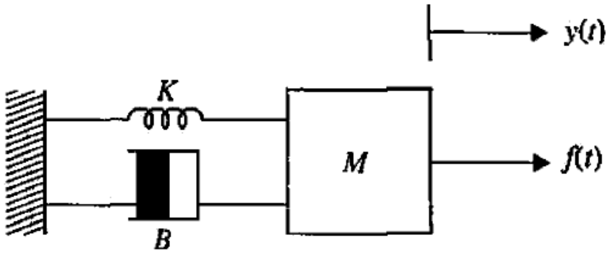
- $t \in \mathbb{R}$ denotes time
- $\mathbf{x} \in \mathbb{R}^n$ denotes the state vector
- $\mathbf{u} \in \mathbb{R}^p$ denotes the input vector
- $\mathbf{y} \in \mathbb{R}^m$ denotes the output vector

- $\mathbf{A} \in \mathbb{R}^{n\times n}$ denotes the system dynamic matrix
- $\mathbf{B} \in \mathbb{R}^{n\times p}$ denotes the input matrix
- $\mathbf{C} \in \mathbb{R}^{m\times n}$ denotes the output or sensor matrix
- $\mathbf{D} \in \mathbb{R}^{m\times p}$ denotes the feedthrough matrix

- For LTI systems, the matrices A,B,C and D are all constant, i.e. not f(*t*)
- For time variant systems → A(*t*), B(*t*), C(*t*), D(*t*)

State Space Equations

Problem 1: Find the SS formulation for the mass-spring-damper system with input *f* (applied force) and output *y* (displacement)



$$\boxed{M\ddot{y} + B\dot{y} + Ky = f}$$

Answer

$$\boxed{\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} f$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State Space Equations

Problem 2: Find the SS formulation for the following system whose i/o relationship is given by the following Diff. Eq.

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + y = 4\ddot{u} - \dot{u} + 5u$$

Answer

$$\frac{Y(s)}{U(s)} = \frac{4s^2 - s + 5}{s^3 + 2s^2 + 3s + 1}$$

$$= \frac{1}{s^3 + 2s^2 + 3s + 1} \cdot \left(4s^2 - s + 5\right) = \frac{Z(s)}{U(s)} \cdot \frac{Y(s)}{Z(s)}$$

$$x_1 = z$$
$$x_2 = \dot{z}$$
$$x_3 = \ddot{z}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

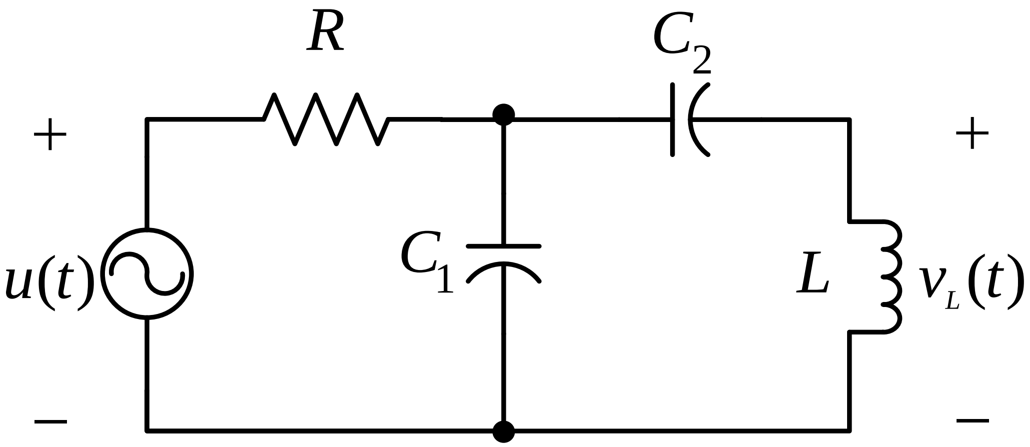
MATLAB

[A B C D] = tf2ss(num,den)

Note that MATLAB uses a flipped state vector assignment

State Space Equations

Problem 3: Find the SS formulation for the following circuit with input *u* (input voltage) and output v_L (inductor voltage)



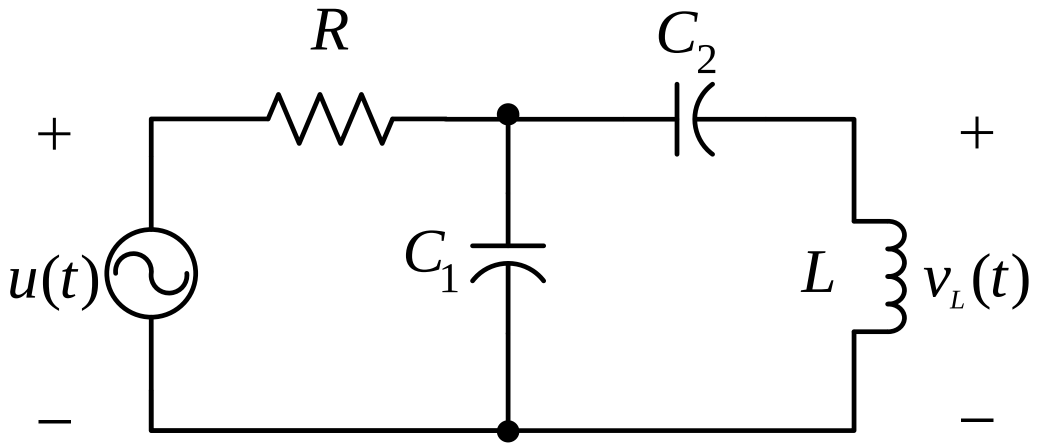
Solution: Use the following state variable assignment

$$\boxed{\begin{aligned} x_1 &= v_{C_1} \\ x_2 &= v_{C_2} \\ x_3 &= i_L \end{aligned}}$$

State Space Equations

Problem 3

$$x_1 = v_{c_1}$$
$$x_2 = v_{c_2}$$
$$x_3 = i_L$$



$$\dot{x}_1 = \dot{v}_{c_1} = \frac{1}{C_1} i_{c_1} = \frac{1}{C_1}\left(i_R - i_L\right) = \frac{1}{C_1}\left(i_R - x_3\right)$$

$$\text{but} \quad u = R i_R + v_{c_1} \rightarrow i_R = -\frac{1}{R} x_1 + \frac{1}{R} u$$

$$\therefore \dot{x}_1 = \frac{1}{C_1}\left(-\frac{1}{R} x_1 + \frac{1}{R} u - x_3\right) = -\frac{1}{RC_1} x_1 - \frac{1}{C_1} x_3 + \frac{1}{RC_1} u$$

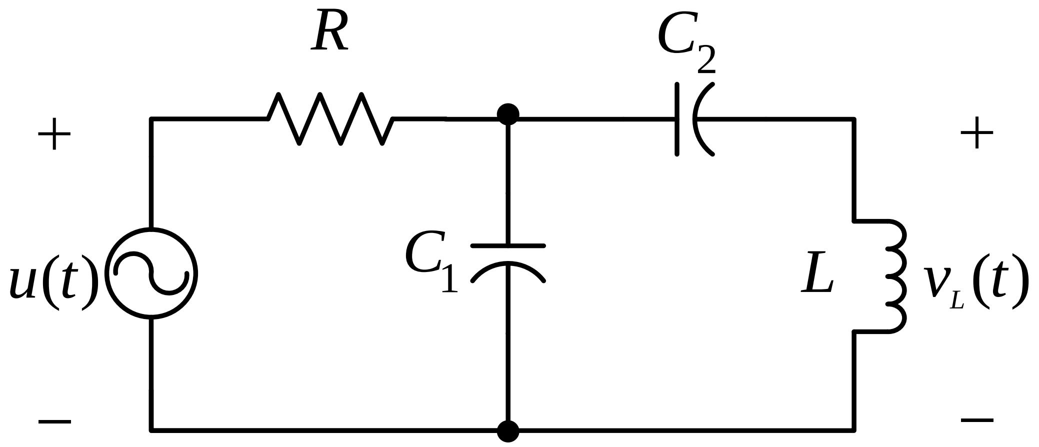
$$\dot{x}_2 = \dot{v}_{c_2} = \frac{1}{C_2} x_3$$

$$\dot{x}_3 = \dot{i}_L = \frac{1}{L} v_L = \frac{1}{L}\left(v_{c_1} - v_{c_2}\right) = \frac{1}{L}\left(x_1 - x_2\right)$$

State Space Equations

Problem 3

$$x_1 = v_{c_1}$$
$$x_2 = v_{c_2}$$
$$x_3 = i_L$$



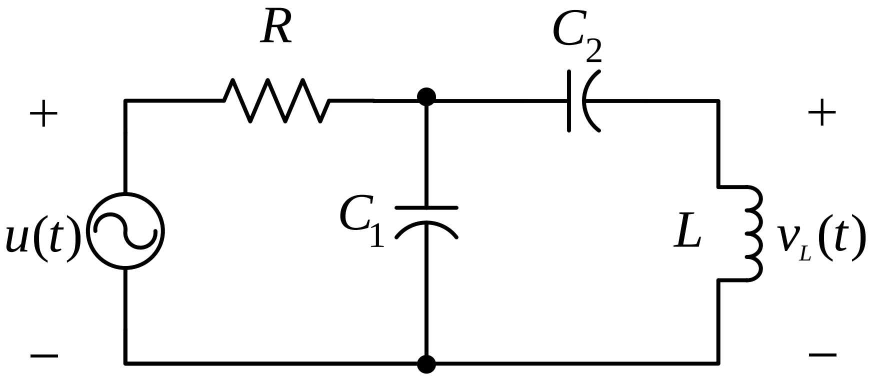
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$v_L = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Is this the only possible SS representation of the circuit?!!

State Space Equations

Problem 3



What if we find $\frac{V_L(s)}{U(s)}$ from cct.?

$$\frac{V_L(s)}{U(s)} = \frac{s^2 C_2 L}{s^3 C_1 C_2 R L + s^2 C_2 L + sR(C_1 + C_2) + 1}$$

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{C_1 C_2 R L} & -\frac{C_1 + C_2}{C_1 C_2 L} & -\frac{1}{R C_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$v_L = \begin{bmatrix} 0 & 0 & \frac{1}{R C_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u$$

Another equivalent SS representation (Same Transfer Function)

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Homogeneous Solution

When the input is zero and the system is only driven by the initial state variables **x**(0)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Taking the Laplace transform (assume **A** is constant for LTI system)

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}\mathbf{x}(0)$$

Solution of SS Equations (Time response)

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s}\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)^{-1}$$

$$\because\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)\left(\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots\right)=\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots-\frac{\mathbf{A}}{s}-\frac{\mathbf{A}^2}{s^2}-\frac{\mathbf{A}^3}{s^3}=\mathbf{I}$$

$$\therefore\left(\mathbf{I}-\frac{\mathbf{A}}{s}\right)^{-1}=\mathbf{I}+\frac{\mathbf{A}}{s}+\frac{\mathbf{A}^2}{s^2}+\frac{\mathbf{A}^3}{s^3}+\ldots$$

Think Maclaurin Expansion for scalars!!

$$\therefore \mathbf{x}(t)=\mathcal{L}^{-1}\left\{\frac{\mathbf{I}}{s}+\frac{\mathbf{A}}{s^2}+\frac{\mathbf{A}^2}{s^3}+\frac{\mathbf{A}^3}{s^4}+\ldots\right\}\mathbf{x}(0)$$

$$\therefore \mathbf{x}(t)=\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2!}+\mathbf{A}^3\frac{t^3}{3!}+\ldots\right)\mathbf{x}(0)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{1}{n!}t^n}$$

Solution of SS Equations (Time response)

$$\therefore \mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \mathbf{A}^3 \frac{t^3}{3!} + \ldots \right) \mathbf{x}(0)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

$$\boxed{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)}$$

State Transition Matrix $\boxed{\mathbf{\Phi}(t) = e^{\mathbf{A}t}}$

It gives the updated state variables at time *t* given the initial state variables **x**(0)

Matrix Exponential
in MATLAB
expm(A)

Solution of SS Equations (Time response)

Property of Matrix exponential

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{(\mathbf{A}+\mathbf{B})} \quad \text{iff} \quad \mathbf{AB}=\mathbf{BA}$$

$$\text{otherwise} \quad e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{(\mathbf{A}+\mathbf{B})}$$

Try to prove it!!

Properties of State Transition Matrix

1) $\mathbf{\Phi}(t_1) = e^{\mathbf{A}t_1}, \quad \mathbf{\Phi}(t_2) = e^{\mathbf{A}t_2}$

$$\mathbf{\Phi}(t_1)\cdot\mathbf{\Phi}(t_2) = e^{\mathbf{A}t_1}\cdot e^{\mathbf{A}t_2} = e^{\mathbf{A}(t_1+t_2)}$$

The second equality holds since $\mathbf{A}t_1$ commutes with $\mathbf{A}t_2$

$$\text{if} \quad t_1 = -t_2 = t \quad \Rightarrow \quad \mathbf{\Phi}(t)\cdot\mathbf{\Phi}(-t) = e^{\mathbf{A}t}\cdot e^{-\mathbf{A}t} = e^{A\cdot 0} = \mathbf{I}$$

$$\therefore \mathbf{\Phi}(-t) \quad \text{is the inverse of} \quad \mathbf{\Phi}(t) \quad \Rightarrow \quad \boxed{\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t)}$$

Solution of SS Equations (Time response)

Properties of State Transition Matrix

2) $\because \mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{x}(0) = e^{\mathbf{A}t_0}\mathbf{x}(0)$

$$\therefore \mathbf{x}(0) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$$

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) = \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0)}$$

3) $\boxed{\mathbf{\Phi}(t_2 - t_0) = \mathbf{\Phi}(t_2 - t_1)\cdot\mathbf{\Phi}(t_1 - t_0)}$

Solution of SS Equations (Time response)

Example

Compute $e^{\mathbf{A}t}$ if $\mathbf{A}=\begin{bmatrix}4 & -5\\ 2 & -3\end{bmatrix}$

$$e^{\mathbf{A}t}=\mathcal{L}^{-1}\left\{\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\right\}$$

$$\left(s\mathbf{I}-\mathbf{A}\right)=\begin{bmatrix}s-4 & 5\\ -2 & s+3\end{bmatrix}$$

$$\left(s\mathbf{I}-\mathbf{A}\right)^{-1}=\frac{1}{\left(s-4\right)\left(s+3\right)+10}\begin{bmatrix}s+3 & -5\\ 2 & s-4\end{bmatrix}$$

$$=\frac{1}{s^2-s-2}\begin{bmatrix}s+3 & -5\\ 2 & s-4\end{bmatrix}=\begin{bmatrix}\dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)}\\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)}\end{bmatrix}$$

Solution of SS Equations (Time response)

Example

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+3}{(s-2)(s+1)} & \dfrac{-5}{(s-2)(s+1)} \\ \dfrac{2}{(s-2)(s+1)} & \dfrac{s-4}{(s-2)(s+1)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{5/3}{s-2} - \dfrac{2/3}{s+1} & \dfrac{-5/3}{s-2} + \dfrac{5/3}{s+1} \\ \dfrac{2/3}{s-2} - \dfrac{2/3}{s+1} & \dfrac{-2/3}{s-2} + \dfrac{5/3}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t} + \frac{5}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{5}{3}e^{-t} \end{bmatrix}$$

MATLAB

```
syms t
A = [4 -5;2 -3]
expm(A*t)
```

Solution of SS Equations (Time response)

Problem

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Solution of SS Equations (Time response)

Problem

Compute $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Forced Solution

When the input **u**(t) is non-zero

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Taking the Laplace transform (assume **A** is constant for LTI system)

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}(0) + \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)\right\}$$

Solution of SS Equations (Time response)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Forced Solution

When the input **u**(t) is non-zero

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\right\}\mathbf{x}(0) + \mathcal{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(s)\right\}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Homogenous solution

Solution of SS Equations (Time response)

Example

Solve the following SS equations, i.e. find **x**(t), if $\mathbf{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}$ and the input u(t) is a unit step function

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}4 & -5\\2 & -3\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\1\end{bmatrix}\mathbf{u}(t)$$

Solution

From previous example with the same **A**, we found the state transition matrix

$$e^{\mathbf{A}t}=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t}+\frac{5}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{5}{3}e^{-t}\end{bmatrix}$$

<u>Substitute in</u>

$$\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Solution of SS Equations (Time response)

$$\mathbf{x}(t)=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{5}{3}e^{2t}+\frac{5}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{5}{3}e^{-t}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$+\int_0^t\begin{bmatrix}\frac{5}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)} & -\frac{5}{3}e^{2(t-\tau)}+\frac{5}{3}e^{-(t-\tau)}\\ \frac{2}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)} & -\frac{2}{3}e^{2(t-\tau)}+\frac{5}{3}e^{-(t-\tau)}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\cdot 1d\tau$$

$$\mathbf{x}(t)=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t}\end{bmatrix}+\int_0^t\begin{bmatrix}\frac{5}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)}\\ \frac{2}{3}e^{2(t-\tau)}-\frac{2}{3}e^{-(t-\tau)}\end{bmatrix}d\tau$$

Solution of SS Equations (Time response)

$$\mathbf{x}(t)=\begin{bmatrix}\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t}\end{bmatrix}+\begin{bmatrix}\frac{5}{6}\left(e^{2t}-1\right)-\frac{2}{3}\left(1-e^{-t}\right)\\ \frac{1}{3}\left(e^{2t}-1\right)-\frac{2}{3}\left(1-e^{-t}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{2}e^{2t}-\frac{3}{2}\\ e^{2t}-1\end{bmatrix}$$

Solution of SS Equations (Time response)

Problem

Repeat computing $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ in an easier way (use eigen decomposition to diagonalize **A**

Hint

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \ldots$$

$$= \mathbf{Q}\mathbf{Q}^{-1} + \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}t + \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}\frac{t^2}{2!} + \ldots.$$

$$= \mathbf{Q}\mathbf{Q}^{-1} + \mathbf{Q}\boldsymbol{\Lambda}t\mathbf{Q}^{-1} + \mathbf{Q}\boldsymbol{\Lambda}^2\frac{t^2}{2!}\mathbf{Q}^{-1} + \ldots.$$

$$= \mathbf{Q}e^{\boldsymbol{\Lambda}t}\mathbf{Q}^{-1}$$

$$= \mathbf{Q}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\mathbf{Q}^{-1}$$

Q: Eigen vector matrix
Λ: diagonal matrix with eigen vaues on main diagonal