Control Systems And Their Components (EE391)

Lec. 8: Open loop SS Control

Thu. April 14th, 2016

Dr. Mohamed Hamdy Osman

Lecture Outline

- Controllability continued
- Observability concept and mathematical condition
- Open loop SS control (no feedback)
- Obtaining least-norm input using the method of Lagrange multipliers

Controllability mathematical condition

Note

- When $\mathbf{C}_k$ is fat, i.e. when $k>n$, you do not need to check the rank of C_k but you can only check the rank of C_n (why?)

$$C_k = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} & \mathbf{A}^n\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix}$$

Check only these columns because the rest of the columns will be dependent on them (why?)

- Because from Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation and hence $\mathbf{A}^n\mathbf{B}$ will depend on previous columns

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0 = 0$$

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + a_{n-2}\mathbf{A}^{n-2} + \ldots + a_1\mathbf{A} + a_0\mathbf{I} = 0$$

$$\therefore \mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - a_{n-2}\mathbf{A}^{n-2} - \ldots - a_1\mathbf{A} + a_0\mathbf{I}$$

Controllability mathematical condition

Summary

- A system with matrices **A**,**B** is said to be controllable if its controllability matrix is full rank (same for discrete and continuous)

$$\operatorname{rank}\{\mathbf{C}_n\} = n$$

$$\mathbf{C}_n = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Final Note

- What if initial state vector is not zero?

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \mathbf{C}_k \mathbf{U} \quad \Rightarrow \boxed{\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \mathbf{C}_k \mathbf{U}}$$

- Condition of controllability stays the same since going from non-zero **x**(0) to **x**(**k**) is just equivalent to going from zero initial state vector to $\mathbf{x}(k)-\mathbf{A}^k\mathbf{x}(0)$

Observability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix}\mathbf{x}(t)$$

- Eigenvalues of **A** are -1, -2 (poles of the system)
- Let's find the TF

$$TF = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{6}{s+1}$$

Same TF as example in slide 12
Where is the other pole at -2 ?

Observability concept

Illustrative Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{x}(t)$$

- Mathematically, the other pole at -2 got canceled because of the 0 in **C** together with **A** being diagonal which basically means that the dynamics of the second state <u>cannot be observed</u> at the output or we say x_2 is not observable
- If you dig deep, you can discover what happened to the eigenvalue at -2 and why it disappeared in TF
- It is because the system has a zero also at -2 that got canceled with the pole at -2 (How can you check zeros??)

$$\begin{vmatrix} z_0\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = 0$$

Observability definition

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \mathbf{B}\mathbf{u}(j)$$

Output Equation

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

$$= \mathbf{C}\mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-j} \mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

Observability Definition

- The system is said to be *observable* if I can uniquely know the initial state variables with the knowledge of the succession of inputs and outputs over finite period of time
- Very important concept as it will be related to State observers that will estimate the state variables from the knowledge of input and output

Observability mathematical condition

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

$$= \mathbf{C}\mathbf{A}^{k}\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k)$$

$$\mathbf{y}(0) = \mathbf{C}\mathbf{x}(0) + \mathbf{D}\mathbf{u}(0)$$

$$\mathbf{y}(1) = \mathbf{C}\mathbf{A}\mathbf{x}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(0) + \mathbf{D}\mathbf{u}(1)$$

$$\mathbf{y}(2) = \mathbf{C}\mathbf{A}^{2}\mathbf{x}(0) + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{C}\mathbf{B}\mathbf{u}(1) + \mathbf{D}\mathbf{u}(2)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Observability mathematical condition

$$\mathbf{y}(0) = \mathbf{Cx}(0) + \mathbf{Du}(0)$$

$$\mathbf{y}(1) = \mathbf{CAx}(0) + \mathbf{CBu}(0) + \mathbf{Du}(1)$$

$$\mathbf{y}(2) = \mathbf{CA}^2\mathbf{x}(0) + \mathbf{CABu}(0) + \mathbf{CBu}(1) + \mathbf{Du}(2)$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \mathbf{y}(2) \\ \vdots \\ \mathbf{y}(k-1) \end{bmatrix}}_{mk\times 1} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{k-1} \end{bmatrix}}_{mk\times n} \underbrace{\mathbf{x}(0)}_{n\times 1} + \underbrace{\begin{bmatrix} \mathbf{D} & 0 & 0 & 0 & \cdots & 0 \\ \mathbf{CB} & \mathbf{D} & 0 & 0 & \cdots & 0 \\ \mathbf{CAB} & \mathbf{CB} & \mathbf{D} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{CA}^{k-2}\mathbf{B} & \mathbf{CA}^{k-3}\mathbf{B} & \cdots & \mathbf{CAB} & \mathbf{CB} & \mathbf{D} \end{bmatrix}}_{mk\times pk} \underbrace{\begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \mathbf{u}(2) \\ \vdots \\ \mathbf{u}(k-1) \end{bmatrix}}_{pk\times 1}$$

$$\boxed{\mathbf{Y} = \mathbf{O}_k\,\mathbf{x}(0) + \mathbf{V}_k\,\mathbf{U}}$$

Observability mathematical condition

$$\mathbf{Y} = \mathbf{O}_k\,\mathbf{x}(0) + \mathbf{V}_k\,\mathbf{U}$$
$$\mathbf{O}_k\,\mathbf{x}(0) = \mathbf{Y} - \mathbf{V}_k\,\mathbf{U}$$

- If I know the inputs and outputs, I know the right hand side of the above equation
- $\mathbf{x}(0)$ is uniquely defined only if rank$\{\mathbf{O}_k\} = n$ (Why?)
- If $\mathbf{O}_k$ is rank deficient then its nullspace is not empty → say $\mathbf{v} \in N\left(\mathbf{O}_k\right)$

$$\mathbf{O}_k\,\mathbf{x}(0) = \mathbf{O}_k\left[\mathbf{x}(0) + \mathbf{v}\right] = \mathbf{Y} - \mathbf{V}_k\,\mathbf{U}$$

- If $\mathbf{O}_k$ is a full rank matrix, its nullspace is empty other than zero vector hence if LHS is known, $\mathbf{x}(0)$ is uniquely determined
- Usually $\mathbf{O}_k$ is a tall matrix

Observability mathematical condition

Summary

- A system with matrices **A**,**C** is said to be observable if its observability matrix is full rank (check only rank of $\mathbf{O}_n$ if $k>n$)

$$\text{rank}\left\{\mathbf{O}_n\right\} = n$$

$$\mathbf{O}_n = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

Minimal realization

Illustrative Example

$$\dot{\mathbf{x}}(t) = -2\mathbf{x}(t) + 3u(t)$$

$$y(t) = 2\mathbf{x}(t)$$

$$\begin{aligned} TF &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \\ &= 2\cdot\frac{1}{s+2}\cdot 3 \\ &= \frac{6}{s+1} \end{aligned}$$

Same TF as example in slides 12 and 20

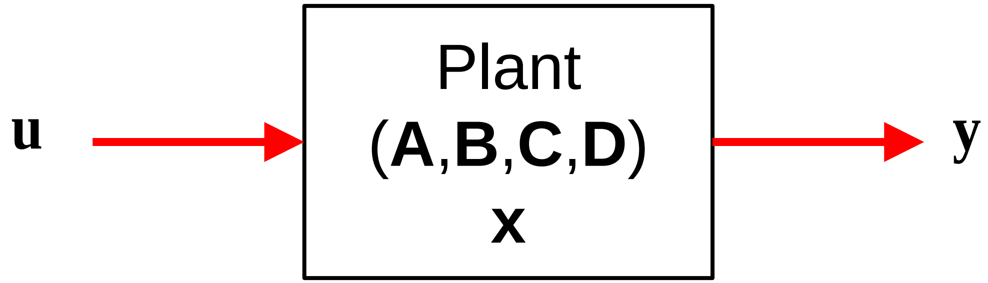
- Both previous examples led to the same TF but one was uncontrollable and the second was unobservable
- This realization of the same TF is both controllable and observable because it is the **minimal realization** (only 1 state variable not 2)

A minimal realization is both controllable and observable (without proof)

Open loop SS control

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$



- The problem here is to find **u** that achieves a certain response **y**
- In SS language, this is exactly equivalent to finding **u** that gives a certain **x** which in turns gives the desired **y**
- More specifically, we would like to reach a certain destination state vector at time *k*, $\mathbf{x}(k) = X_{des}$ and the problem is to find $\mathbf{u}(k)$ for $k = 0$ to $k-1$ that steers the system from $\mathbf{x}(0)$ to X_{des}
- Clearly this is open loop control since no feedback is used.
- The disadvantage is that we assume perfect knowledge of A,B,C,D and noise free operation

Open loop SS control

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \mathbf{B}\mathbf{u}(j)$$

$$\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \mathbf{B}\mathbf{u}(j)$$

n×1

$$= \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}(k-1) \\ \mathbf{u}(k-2) \\ \mathbf{u}(k-3) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

kp×1

n×*kp*

Controllability matrix $\mathbf{C}_k$

U

concatenation of k input vectors each is p×1

$$\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \mathbf{C}_k \mathbf{U}$$

Open loop SS control

$$\mathbf{x}(k) - \mathbf{A}^k \mathbf{x}(0) = \mathbf{C}_k \mathbf{U}$$

$$\text{put} \quad \mathbf{x}(k) = \mathbf{X}_{des}$$

$$\mathbf{X}_{des} - \mathbf{A}^k \mathbf{x}(0) = \mathbf{C}_k \mathbf{U}$$

Without loss of generality, assume zero initial state vector, **x**(0) = 0

$$\underset{n\times 1}{\mathbf{X}_{des}} = \underset{n\times kp}{\mathbf{C}_k} \underset{kp\times 1}{\mathbf{U}}$$

- Assume system is controllable (**C**$_k$ is full row rank) → will see why
- The goal is to find **U** to reach **X**$_{des}$. However, there are infinite solutions to the above equation (why??) → because number of unknowns > number of equations
- We need to impose a constraint to obtain a unique solution (What is the interesting solution we are looking for?)
- We will find the input with minimum energy (least-norm)

Least-norm input

$$\underset{n\times 1}{\mathbf{X}_{des}} = \underset{n\times kp}{\mathbf{C}_k}\, \underset{kp\times 1}{\mathbf{U}}$$

- We will find the input with minimum energy (least-norm)
- Energy is the squared norm of the input obtained as

$$E = \mathbf{U}^{\mathbf{T}}\mathbf{U}$$

- The final problem we will solve is formulated as

$$\text{minimize} \quad \mathbf{U}^{\mathbf{T}}\mathbf{U} \quad \text{subject to} \quad \mathbf{X}_{des} = \mathbf{C}_k\,\mathbf{U}$$

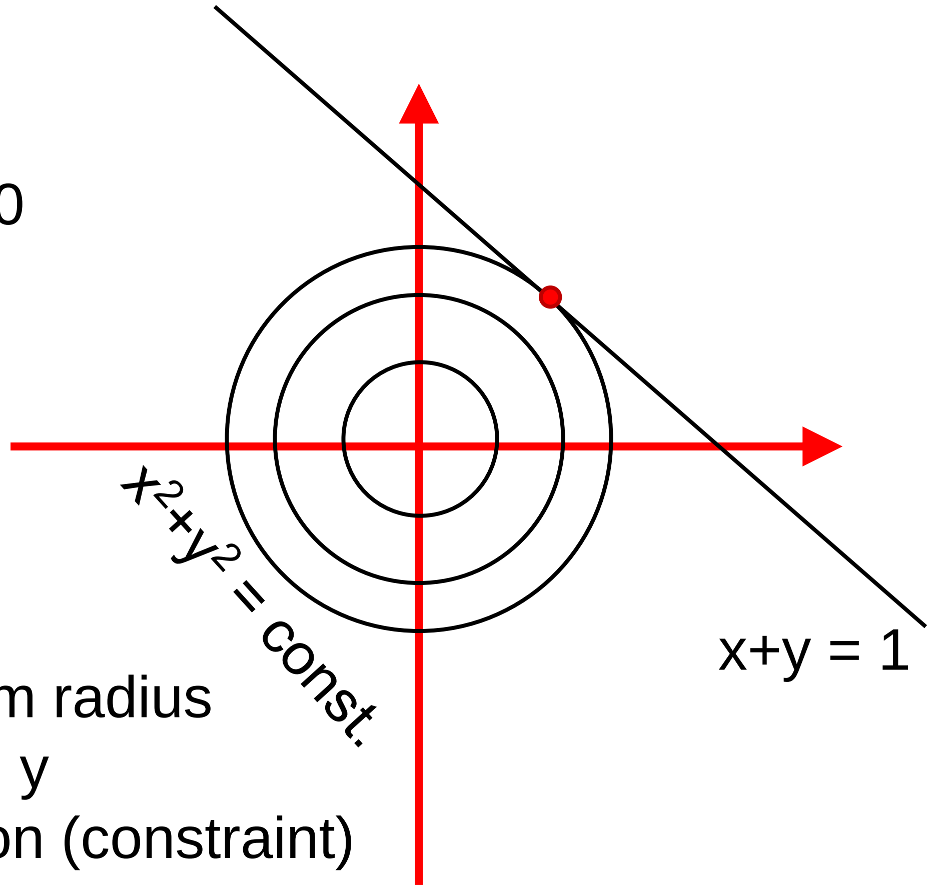
- We will use a method called Lagrange multipliers to do this constrained optimization problem

Method of Lagrange multipliers

Illustrative Example

- Minimize x^2+y^2 subject to the constraint $x+y = 1$

Graphically



<u>If there is no constraint</u>

$\min(x^2+y^2) = 0$ at $x = 0$ and $y = 0$

<u>Under the constraint</u>

We will increase radius of circle until it becomes tangent to the line $x+y = 1$

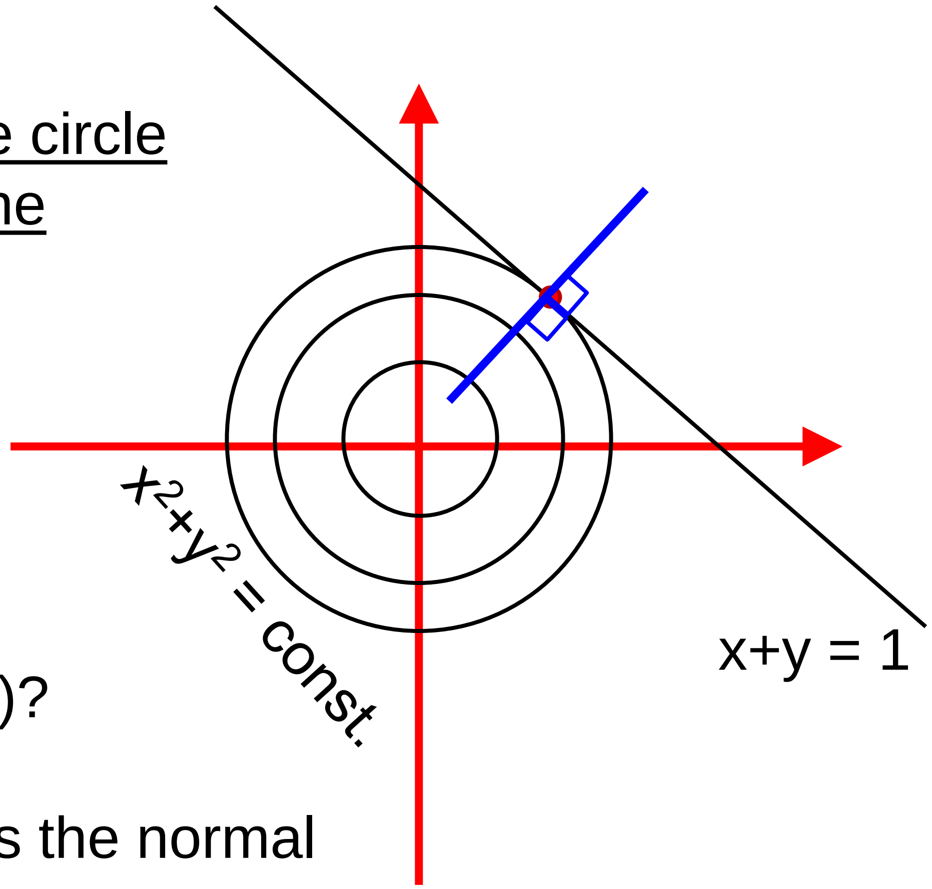
This corresponds to the minimum radius of the circle at which point x and y will also satisfy the line's equation (constraint)

Method of Lagrange multipliers

Illustrative Example

- Minimize x^2+y^2 subject to the constraint $x+y = 1$

Graphically



At the solution, the normal to the circle is parallel to the normal of the line

How do we obtain normal to any function (can be Contour in 2D or surface in 3D Or anything in higher dimension)?

→ **Gradient** of the function gives the normal

Method of Lagrange multipliers

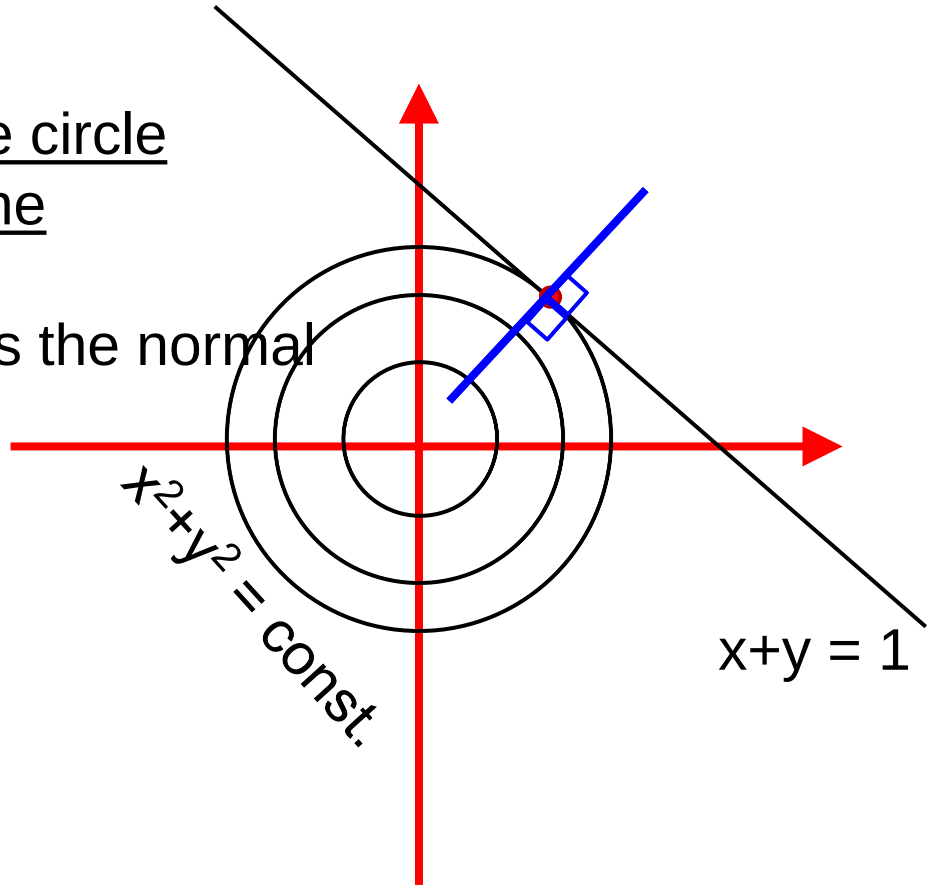
Illustrative Example

- Minimize x^2+y^2 subject to the constraint $x+y = 1$

Graphically

<u>At the solution, the normal to the circle is parallel to the normal of the line</u>

→ **<u>Gradient</u>** of the function gives the normal



$$\nabla\left(x^2+y^2\right)=\lambda\nabla\left(x+y\right)$$

Nabla operator

Scaling factor is Lagrange multiplier

Method of Lagrange multipliers

Illustrative Example

- Minimize x^2+y^2 subject to the constraint $x+y = 1$

$$\nabla\left(x^2+y^2\right)=\lambda\nabla\left(x+y\right)$$

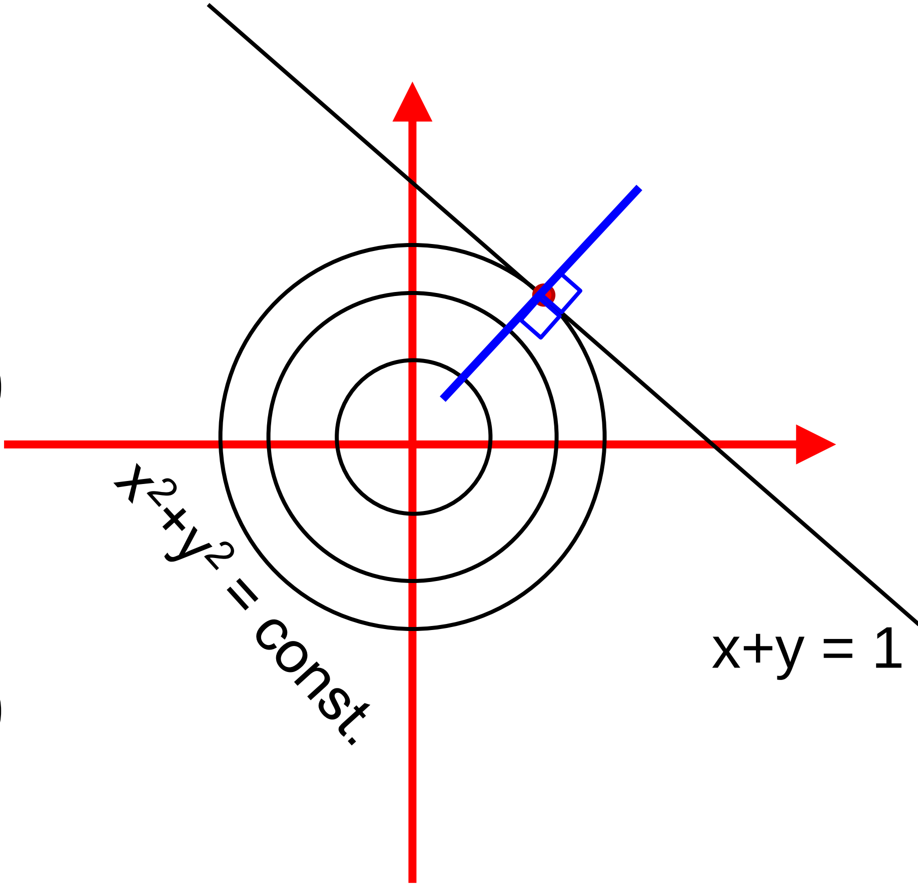
$$\nabla\left(x^2+y^2-\lambda\left(x+y\right)\right)=0$$

$$\Rightarrow\frac{\partial}{\partial x}\left(x^2+y^2-\lambda\left(x+y\right)\right)=0$$

$$2x-\lambda=0 \quad \Rightarrow \quad \boxed{x=\lambda/2}$$

$$\Rightarrow\frac{\partial}{\partial y}\left(x^2+y^2-\lambda\left(x+y\right)\right)=0$$

$$2y-\lambda=0 \quad \Rightarrow \quad \boxed{y=\lambda/2}$$



Method of Lagrange multipliers

Illustrative Example

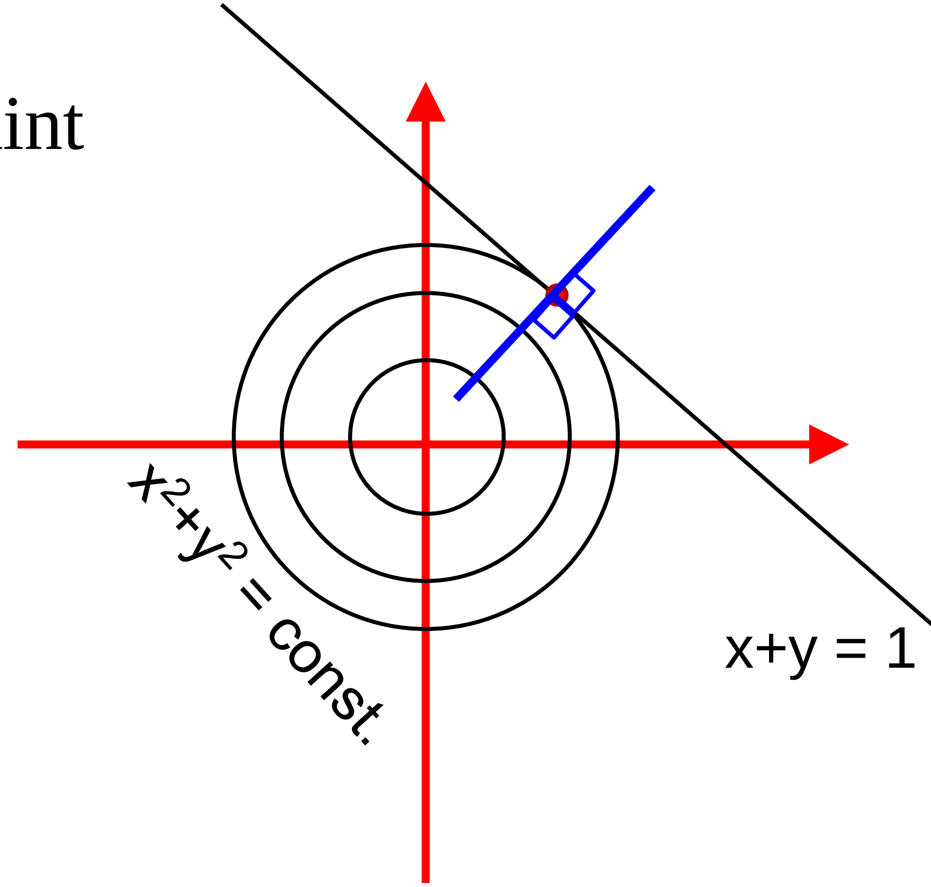
- Minimize x^2+y^2 subject to the constraint $x+y = 1$

Then find λ from the constraint

$$x + y = 1$$

$$\lambda/2 + \lambda/2 = 1 \quad \Rightarrow \quad \lambda = 1$$

$$\therefore \boxed{x = 1/2, y = 1/2}$$



Method of Lagrange multipliers

Generally speaking

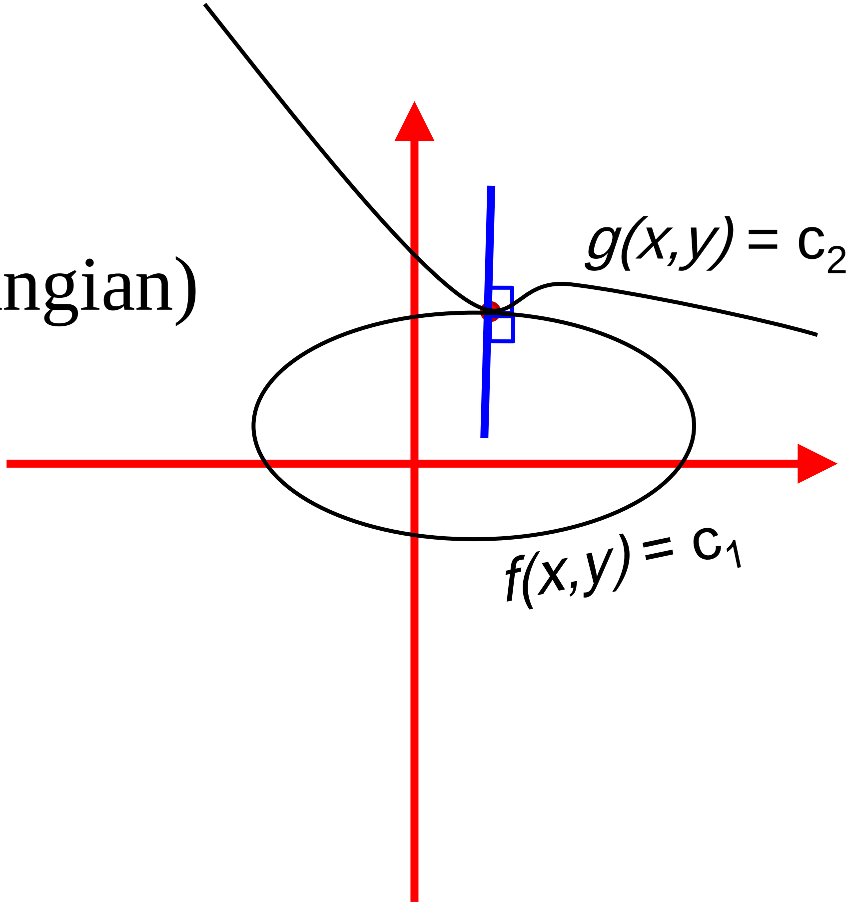
- Minimize $f(x,y) = c_1$ subject to the constraint $g(x,y) = c_2$

$$\nabla(f - \lambda g) = 0$$

$\nabla J = 0$ (J is called Lagrangian)

Find variables in terms of λ

Then λ find from constraint



Least-norm input

- Back to our problem we want to solve

$$\text{minimize} \quad \mathbf{U}^{\mathbf{T}}\mathbf{U} \quad \text{subject to} \quad \mathbf{X}_{des} = \mathbf{C}_k\,\mathbf{U}$$

Take care that we have *n* constraints inside X_{des} not just one

Lagrangian must be a real scalar

$$J = \mathbf{U}^{\mathbf{T}}\mathbf{U} - \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}$$

1×1 (J), 1×1 ($\mathbf{U}^{\mathbf{T}}\mathbf{U}$), $1\times n$ ($\boldsymbol{\lambda}^{\mathbf{T}}$), $n\times1$ ($\mathbf{C}_k\,\mathbf{U}$)

So we must have *n* Lagrange multipliers in vector **λ**

$$\nabla J = 0$$

$$\nabla_{\mathbf{u}}\left(\mathbf{U}^{\mathbf{T}}\mathbf{U} - \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}\right) = 0$$

Least-norm input

$$\nabla J = 0$$

$$\nabla_{\mathbf{u}} \left(\mathbf{U}^{\mathbf{T}} \mathbf{U} - \boldsymbol{\lambda}^{\mathbf{T}} \mathbf{C}_k \mathbf{U} \right) = 0$$

When we do partial derivates, we will differentiate with each element inside vector **u** and concatenate the results

Least-norm input

$$\mathbf{U}^{\mathbf{T}}\mathbf{U} = u_1^2(0) + u_2^2(0) + \ldots + u_p^2(0) + \ldots + u_1^2(k-1) + u_2^2(k-1) + \ldots + u_p^2(k-1)$$

$$\nabla_{\mathbf{u}}\left(\mathbf{U}^{\mathbf{T}}\mathbf{U}\right) = \begin{bmatrix} \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_1(k-1)} \\ \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_2(k-1)} \\ \vdots \\ \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_p(k-1)} \\ \vdots \\ \vdots \\ \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_1(0)} \\ \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_2(0)} \\ \vdots \\ \dfrac{\partial \mathbf{U}^{\mathbf{T}}\mathbf{U}}{du_p(0)} \end{bmatrix} = \begin{bmatrix} 2u_1(k-1) \\ 2u_2(k-1) \\ \vdots \\ 2u_p(k-1) \\ \vdots \\ \vdots \\ 2u_1(0) \\ 2u_2(0) \\ \vdots \\ 2u_p(0) \end{bmatrix} = 2\mathbf{U}$$

$$\Rightarrow \quad \boxed{\nabla_{\mathbf{u}}\left(\mathbf{U}^{\mathbf{T}}\mathbf{U}\right) = 2\mathbf{U}}$$

Least-norm input

$$\nabla_{\mathbf{u}}\left(\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}\right)$$

Let $\mathbf{v}^{\mathbf{T}} = \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k \quad \Rightarrow \quad \mathbf{v}^{\mathbf{T}}\mathbf{U} = v_1 u_1(k-1) + v_2 u_2(k-1) + \ldots + v_p u_p(k-1) + \ldots$

$$\nabla_{\mathbf{u}}\left(\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}\right) = \nabla_{\mathbf{u}}\left(\mathbf{v}^{\mathbf{T}}\mathbf{U}\right) = \begin{bmatrix} \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_1(k-1)} \\ \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_2(k-1)} \\ \vdots \\ \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_p(k-1)} \\ \vdots \\ \vdots \\ \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_1(0)} \\ \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_2(0)} \\ \vdots \\ \dfrac{\partial \mathbf{v}^{\mathbf{T}}\mathbf{U}}{du_p(0)} \end{bmatrix} = \mathbf{v}$$

$$\Rightarrow \quad \boxed{\nabla_{\mathbf{u}}\left(\boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}\right) = \mathbf{C}_k^{\mathbf{T}}\boldsymbol{\lambda}}$$

Least-norm input

$$\nabla J = 0$$

$$\nabla_{\mathbf{u}}\left(\mathbf{U}^{\mathbf{T}}\mathbf{U} - \boldsymbol{\lambda}^{\mathbf{T}}\mathbf{C}_k\,\mathbf{U}\right) = 0$$

$$2\mathbf{U} - \mathbf{C}_k^{\mathbf{T}}\boldsymbol{\lambda} = 0 \quad \Rightarrow \quad \boxed{\mathbf{U}_{\min} = \frac{1}{2}\mathbf{C}_k^{\mathbf{T}}\boldsymbol{\lambda}}$$

Substitue $\mathbf{U}_{\min}$ in the constraint to get $\boldsymbol{\lambda}$

$$\mathbf{C}_k\,\mathbf{U} = \mathbf{X}_{des}$$

$$\frac{1}{2}\mathbf{C}_k\,\mathbf{C}_k^{\mathbf{T}}\boldsymbol{\lambda} = \mathbf{X}_{des} \quad \Rightarrow \quad \boldsymbol{\lambda} = 2\left(\mathbf{C}_k\,\mathbf{C}_k^{\mathbf{T}}\right)^{-1}\mathbf{X}_{des}$$

$$\boxed{\mathbf{U}_{\min} = \mathbf{C}_k^{\mathbf{T}}\left(\mathbf{C}_k\,\mathbf{C}_k^{\mathbf{T}}\right)^{-1}\mathbf{X}_{des}}$$

Least-norm input

$$\mathbf{U}_{\min} = \mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des}$$

Notes

- Above equation gives the least norm input having minimum energy which steers the system from the zero initial state vector to $\mathbf{X}_{des}$
- In order for $\left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1}$ to exist, $\mathbf{C}_k$ must be full rank, i.e. the system

  must be controllable at the first place (think why this condition must be true?)
- If initial state vector was not zero

$$\mathbf{U}_{\min} = \mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \left(\mathbf{X}_{des} - \mathbf{A}^k \mathbf{x}(0) \right)$$

Least-norm input

$$\mathbf{U}_{\min} = \mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des}$$

Notes

- Minimum energy required to reach $\mathbf{X}_{\mathrm{des}}$ starting at zero initial state vector in *k* steps can be finally calculated as

$$E_{\min} = \mathbf{U}_{\min}^{\mathbf{T}} \mathbf{U}_{\min} = \left[\mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des} \right]^{\mathbf{T}} \mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des}$$

$$= \mathbf{X}_{des}^{\mathbf{T}} \left[\left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \right]^{\mathbf{T}} \mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des}$$

$$= \mathbf{X}_{des}^{\mathbf{T}} \left(\mathbf{C}_k \mathbf{C}_k^{\mathbf{T}} \right)^{-1} \mathbf{X}_{des}$$

Least-norm input

Example

Plot E_{min} versus k for the system having

$$\mathbf{A}=\begin{bmatrix}1.75 & 0.8\\ -0.95 & 0\end{bmatrix},\mathbf{B}=\begin{bmatrix}1\\0\end{bmatrix},\mathbf{X}(0)=\begin{bmatrix}0\\0\end{bmatrix},\mathbf{X}_{des}=\begin{bmatrix}1\\1\end{bmatrix}$$

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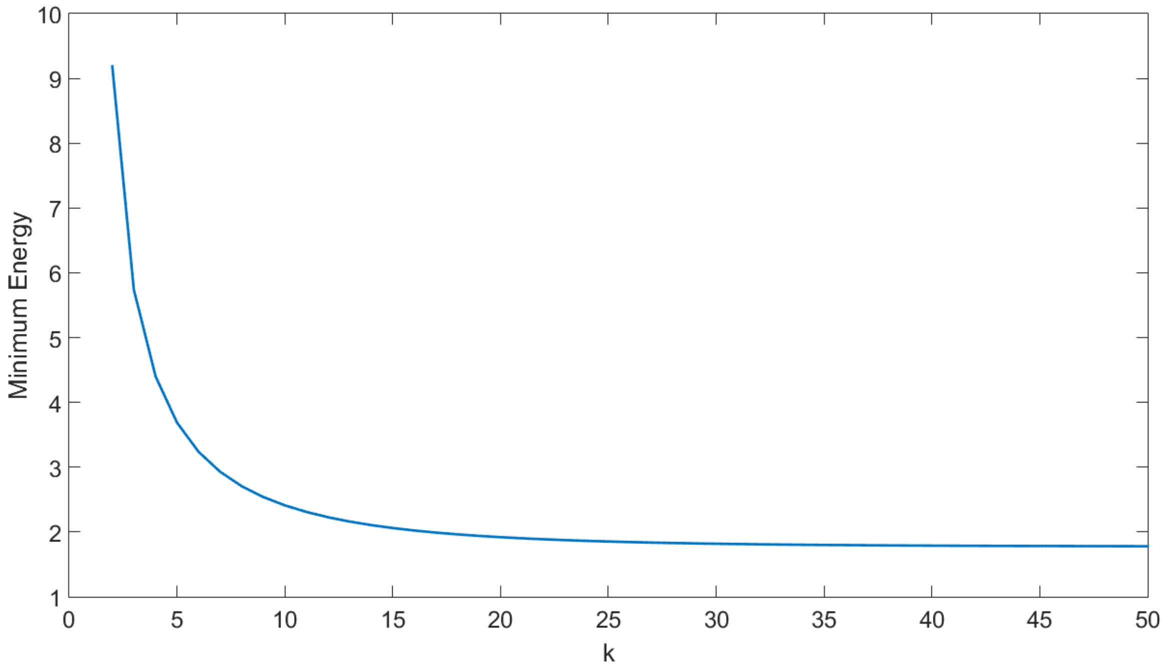
```
clear all
close all

A = [1.75 0.8;-0.95 0];
B = [1;0];
Xdes = [1;1];
Xinitial = [0;0];

% calculate controllability matrices

minimumEnergyVect = zeros(49,1);

for k = 2:50
    Cont = B;
    for m = 1:k-1
        Cont = [Cont A^m*B];
    end
    Umin = Cont.' * inv(Cont*Cont.') * (Xdes-A^k*Xinitial);
    minimumEnergyVect(k-1) = Umin.' * Umin;
end

figure
plot(2:50,minimumEnergyVect)
ylabel('Minimum Energy')
xlabel('k')
```

Least-norm input

Example

Plot E_{min} versus k for the system having



As k increases meaning that I give the system all the time it needs to reach the destination state vector, the minimum energy required to reach $\mathbf{X}_{des}$ decreases until it converges to a steady state value